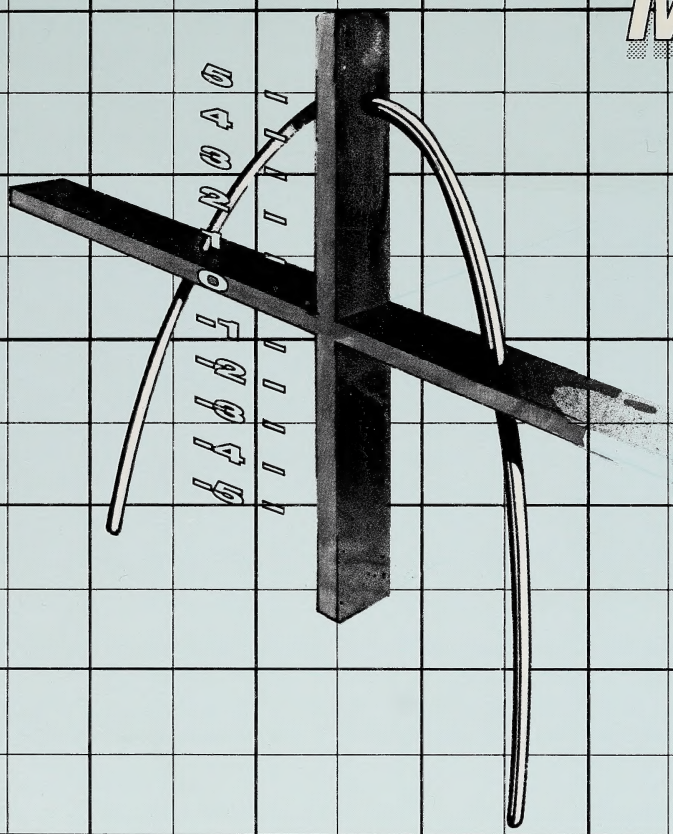
MATHEMATICS 9

QUADRATIC RELATIONS
UNIT 5



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Welcome

Distance Learning

You have chosen an alternate form of learning that allows you to work at your own pace. You will be responsible for your own schedule, for disciplining yourself to study the units thoroughly, and for completing your units regularly. We wish you much success and enjoyment in your studies.

Mathematics 30 Student Module Unit 5 Quadratic Relations Alberta Distance Learning Centre ISBN No. 0-7741-0358-2

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General Information

This information explains the basic layout of each booklet.

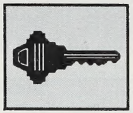
- **What You Already Know** and **Review** are to help you look back at what you have previously studied. The questions are to jog your memory and to prepare you for the learning that is going to happen in this unit.
- As you begin each **Topic**, spend a little time looking over the components. Doing this will give you a preview of what will be covered in the topic and will set your mind in the direction of learning.
- **Exploring the Topic** includes the objectives, concept development, and activities for each objective. Use your own papers to arrive at the answers in the activities.
- To summarize what you have learned, and to find instructions on doing the unit assignment, turn to the **Unit Summary** at the end of the unit.
- The **Appendices** include the solutions to **Activities (Appendix A)** and any other charts, tables, etc. which may be referred to in the topics **(Appendix B, etc.)**.

Visual Cues

Visual cues are pictures that are used to identify important areas of the material. They are found throughout the booklet.

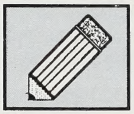
An explanation of what they mean is written beside each visual cue.

What You Already Know
- reviewing what you already know



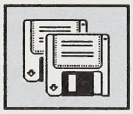
Key Idea
- flagging important ideas

Audiotape
- learning by listening to an audiotape



Review
- studying previous concepts

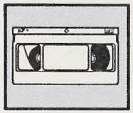
Another View
- exploring different perspectives



Computer Software
- learning by using computer software

Introduction
- introducing the unit

Solutions
- correcting the activities



Videotape
- learning by viewing a videotape

What Lies Ahead
- previewing the unit

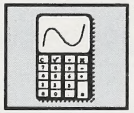
Extra Help
- providing additional study

Print Pathway
- choosing a print alternative

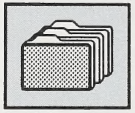
Exploring the Topic
- actively learning new concepts

Extensions
- going on with the topic

Calculator
- using your calculator



Graphing Calculator
- using your graphing calculator



What You Have Learned
- summarizing what you have learned

Mathematics 30

Course Overview

Mathematics 30 contains 7 units. Beside each unit is a percentage that indicates what the unit is worth in relation to the rest of the course. The units and their percentages are listed below. You will be studying the unit that is shaded.

Unit 1 Polynomial Functions	13%
Unit 2 Logarithms	13%
Unit 3 Sequences, Series, Limits	13%
Unit 4 Trigonometry	18%
Unit 5 Quadratic Relations	12%
Unit 6 Statistics	18%
Unit 7 Permutations and Combinations	13%
	100%

Unit Assessment

After completing the unit you will be given a mark based totally on a unit assignment. This assignment will be found in the Assignment Booklet.

Unit Assignment - 100%

If you are working on a CML terminal, your teacher will determine what this assessment will be. It may be

Unit Assignment - 50%
Supervised Unit Test - 50%

Introduction to Quadratic Relations

This unit covers topics dealing with quadratic relations. Each topic contains explanations, examples, and activities to assist you in understanding quadratic relations.

For this unit you will need to have access to a graphing calculator and/or a computer software package on graphing.

You can evaluate your understanding of each topic by working through the activities. Answers are found in the solutions in **Appendix A**. In several cases there is more than one way to do the question.

Unit 5 Quadratic Relations

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	• Introduction • What Lies Ahead • Exploring Topic 1	
40%	**Topic 2:** Describing Conics Through Parameters	19
	• Introduction • What Lies Ahead • Exploring Topic 2	
25%	**Topic 3:** Defining Each Conic as a Locus of Points	79
	• Introduction • What Lies Ahead • Exploring Topic 3	
25%	**Topic 4:** Using Eccentricity to Define Conics	90
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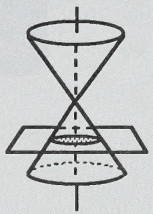
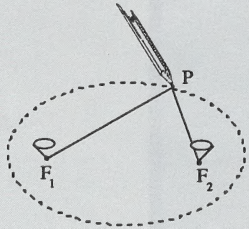
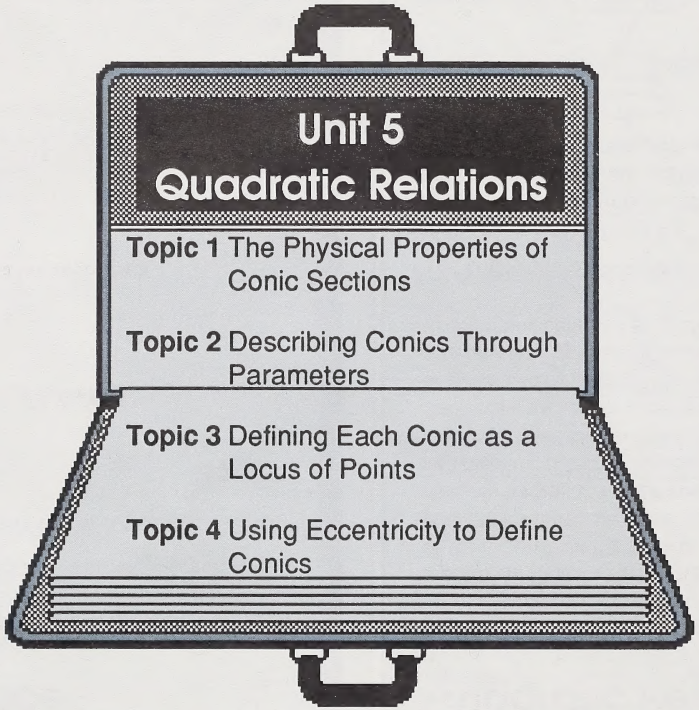
Quadratic Relations

The study of quadratic relations connects several topics from the high school mathematics program. The topics of coordinate geometry and algebra are linked to the analysis of relations and the resulting graphs of the locus of these relations. Because there are a number of practical phenomena that can be described in terms of quadratic relations, it is possible for you to see the connection between real-world applications and theoretical representations. Since all conic sections can be defined as the locus of a point such that the ratio of its distance from a fixed point and a fixed line is constant and each of the four conics can be generated by altering this ratio, the study of quadratic relations provides the opportunity to analyse a complex mathematical relation to determine common properties and structures.

The graphs of the relations defined by quadratic equations in x and y of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ in which A, B, C, D, E, and F are real numbers and neither A, B, nor C equals 0 are called conic sections. A conic section is the intersection of a plane and a double-napped circular cone.






Unit 5
Quadratic Relations
Topic 1 The Physical Properties of Conic Sections
Topic 2 Describing Conics Through Parameters
Topic 3 Defining Each Conic as a Locus of Points
Topic 4 Using Eccentricity to Define Conics

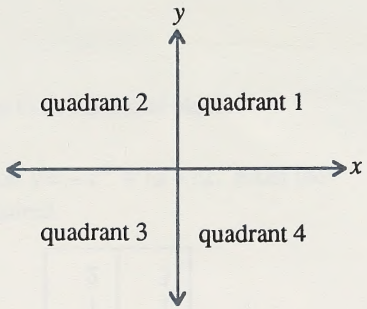
What You Already Know

Refresh your memory!

The following is a review of some important concepts.

- A coordinate plane is divided into four regions.

  In quadrant 1 the x- and y-values are positive.
  In quadrant 2 the x-value is negative and the y-value is positive.
  In quadrant 3 the x- and y-values are negative.
  In quadrant 4 the x-value is positive and the y-value is negative.

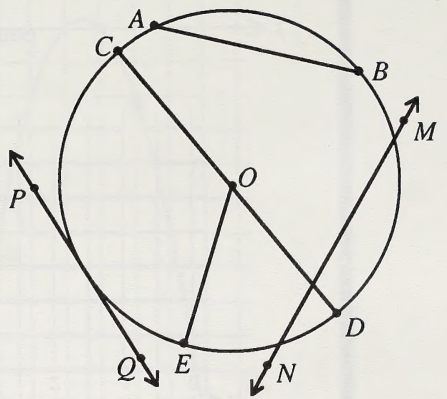


- Parallel lines are lines in the same plane that do not intersect.

- Perpendicular lines are lines in the same plane that intersect at a right angle (90°).

- Parts of a circle include the following:

  O – centre of the circle
  OE – radius
  CD – diameter
  AB – chord or segment
  MN – secant line
  PQ – tangent line



- In the expression $3x + 5y$, 3 is the numerical coefficient of x and 5 is the numerical coefficient of y.

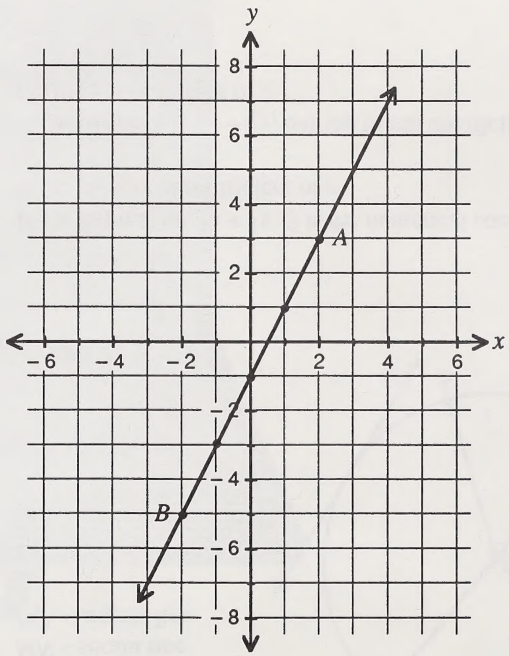
  In the expression $ax + by$, a is the literal coefficient of x and b is the literal coefficient of y.

- The grid with the intersecting pair of axes is called a Cartesian plane. An ordered pair (x, y) names the coordinates of a point on this grid.

  For example, graph the function $y = 2x - 1$ on the grid.

  Step 1: Obtain a table of values for selected points.

x	y
−2	−5
−1	−3
0	−1
1	1
2	3



Step 2: Plot the points, and then join the points. This is the graph of the function.

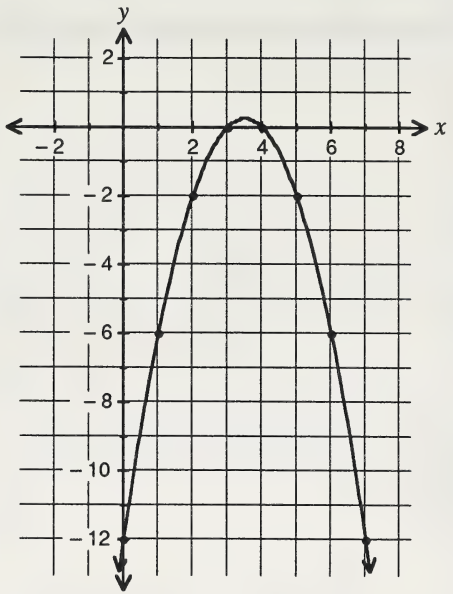
Find the length of segment AB on the previous graph by using the distance formula.

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$= \sqrt{[2-(-2)]^2 + [3-(-5)]^2}$$
$$= \sqrt{4^2 + 8^2}$$
$$= \sqrt{16 + 64}$$
$$= \sqrt{80}$$
$$\doteq 8.94$$

The length is 8.94 units correct to two decimal places.

- Sketch the graph of the function $y = -x^2 + 7x - 12$. From the graph find the information required.

x	y
0	−12
1	−6
2	−2
3	0
4	0
5	−2
6	−6
7	−12



The vertex is $(3.5, 0.25)$.

The axis of symmetry is $x = 3.5$.

The x-intercepts are 3 and 4.

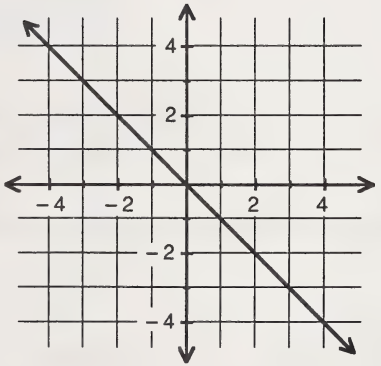
The y-intercept is -12.

The maximum value is 0.25.

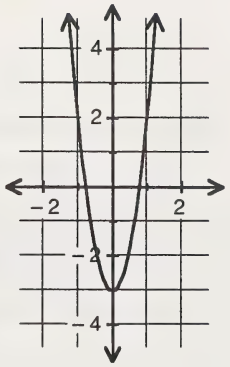
Use a graphing calculator to draw the graph of $y = -x^2 + 7x - 12$. It is the same as the graph that was previously drawn. You should be able to draw this graph and any other similar linear or quadratic function graph. The procedure to do this will depend on which graphing calculator you are using.

- Recall the shapes of the following:

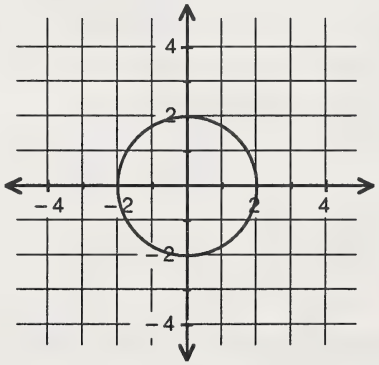
Straight line



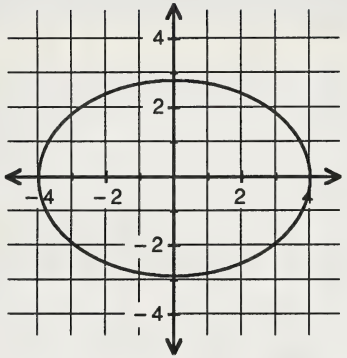
Parabola



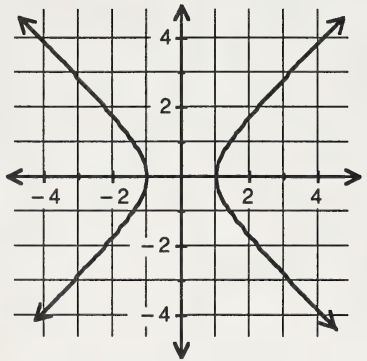
Circle



Ellipse



Hyperbola



- Given the function $y = f(x)$, recall the effect of a in each of the following:

$y = f(x) + a$

The graph moves up or down a units.

$y = f(x + a)$

The graph moves left or right a units.

$y = af(x),\ a > 0$

The graph stretches or compresses vertically if a is positive.

$y = -af(x),\ a > 0$

The graph reflects in the x-axis and stretches or compresses if a is positive.

Review

Try the following review questions.

1. You are given two points $A(-5, 7)$ and $B(6, -5)$ on the Cartesian plane. Find the length of $\overline{AB}$ to two decimal places.

2. Draw the graph for $y = x^2 - 4$ using a table of values. From the graph determine the following.

   a. the coordinates of the vertex
   b. the equation of the axis of symmetry
   c. the x-intercepts
   d. the y-intercept
   e. the maximum or minimum value

3. Draw the graph of $y = x^2 - 4$ on a graphing calculator. Compare this graph with the one obtained in question 2.

4. State in which quadrant each of the following points would be located.

   a. $(-5, -9)$ b. $(4, 3)$

   c. $(-3, 6)$ d. $(5, -9)$

5. Using a graphing calculator, graph the function $y = x^3$. Then graph each of the following functions and describe how each differs from the function $y = x^3$.

   a. $y = x^3 - 5$ b. $y = (x + 5)^3$

   c. $y = -(x^3 - 5)$ d. $y = (-x)^3 - 5$

Now go to the **Review** solutions in **Appendix A.**

Topic 1 The Physical Properties of Conic Sections

Introduction

This topic studies the physical aspects of the conic sections. You will learn how they are formed, how to investigate them, and how to observe them in everyday life.

In this topic you will learn how to cut a double-napped cone to produce the conic sections.

What Lies Ahead

In this topic you will learn to

1. demonstrate an understanding of the physical properties of the conic sections with respect to the intersection of a plane and a double-napped cone.

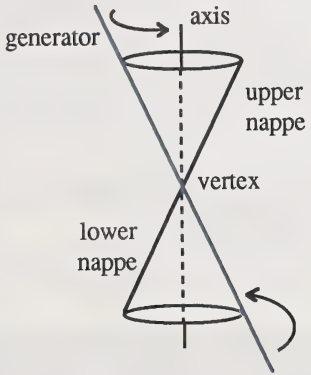
Now that you know what to expect, turn the page to begin your study of the physical properties of conic sections.

Exploring Topic 1

Activity 1

Demonstrate an understanding of the physical properties of the conic sections with respect to the intersection of a plane and a double-napped cone.

A Greek mathematician, Appollonius, who lived about 200 B.C., was the first to discover the shapes that occur when a double-napped cone is intersected by a plane. He pictured the cones as two **nappes** that are connected at their respective vertices and extend away infinitely. The two portions are referred to as the upper nappe and the lower nappe. This is illustrated in the following diagram of a double-napped cone.

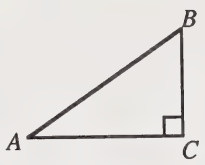


You can also visualize the nappes by rotating an oblique line about a vertical axis. When you do this, the oblique line is referred to as the **generator**.

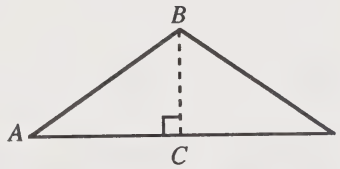
The shapes described by the circle, parabola, ellipse, and hyperbola are often called **conics**.

A conic is obtained when a plane intersects a double cone. The intersection may occur horizontally, vertically, or obliquely (slanted).

Consider the right triangle ABC which follows. You will notice that a cone can be realized if the triangle is revolved about the side BC.

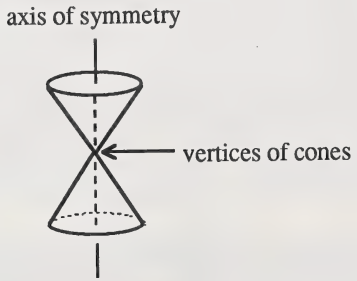


Right ΔABC



A cross section of a single circular cone is produced when ΔABC is rotated about side BC.

The double-napped cone is obtained by placing two single circular cones so that their vertices coincide. The following diagram illustrates a double-napped cone.



When a double-napped cone is intersected by a plane at right angles to its axis, the cross section is a **circle**.

circle

The intersection of one part of the double-napped cone and the plane perpendicular to the axis of symmetry of the cone produces a circle.

If the plane intersects the double-napped cone parallel to a generator, the cross section is a **parabola.**

parabola

The intersection of one nappe of the double-napped cone and the plane parallel to a generator of the cone results in a parabola.

If the plane intersects one nappe of the double-napped cone and if not at right angles to the axis and not parallel to a generator, then the cross section is an **ellipse.**

ellipse

The intersection of one nappe of the double cone and a plane not perpendicular nor parallel to the axis of symmetry of the cone, and not parallel to the generator, results in an ellipse.

When both nappes of a double cone are intersected by a plane (not passing through the vertex and parallel to the axis), the cross section produces a **hyperbola.**

hyperbola

The intersection of both nappes of the double cone and the plane produces a hyperbola.

The shapes that are produced by the intersection of a plane and a double-napped cone are conic sections. Those that have been discussed are called proper or standard conic sections. Others are possible and are referred to as **degenerate** conic sections. The term *degenerate* refers to a change from the regular form.

Degenerate conics occur in three specific cases.

- Consider the case when a horizontal plane approaches the common vertex of the double-napped cone.

  What do you observe at the intersection?

  You should notice that the circle is getting smaller and smaller, and when it gets to the vertex, it becomes a **point**. You can then say the circle becomes a point.

- Another case occurs when a plane parallel to the slant side of the cone intersects the vertex of that cone.

  What do you observe as the the plane gets closer to the vertex?

  You should be able to visualize that the parabolic shape of the intersection is getting larger and larger and ultimately becomes a **straight line**. You can now say that a parabola becomes a straight line.

- There is also the case where a vertical plane intersects the axis of symmetry of the double cone.

  What do you notice as the plane gets closer and closer to the axis?

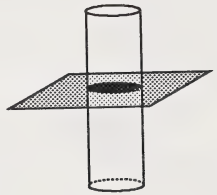
  You should visualize that the two portions of the hyperbolic shapes are getting progressively larger, then become **two straight lines crossing at the vertex**.

You already know what happens when an inclined plane parallel to the generator intersects a cone. An ellipse is formed at the cross section. What happens as the plane becomes perpendicular to the axis?

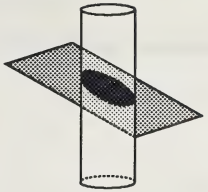
You will notice that the intersection is a **circle**.

Other exceptional degenerate conics are **two parallel lines** and the case of **no locus**.

The last two degenerate cases can be observed if you look at a cylinder. A cylinder (such as a pipe) can be considered as an extreme degenerate case of the cone. If you take a cylinder and cut it with a plane in various ways, you get examples of conics.

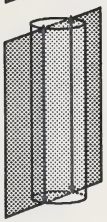


If the plane cuts the cylinder at right angles, you will get a circle.

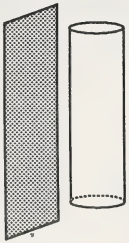


If the plane cuts the cylinder at any other angle less than 90°, **an ellipse** is produced.

If the plane is parallel to the sides of the cylinder and it touches the cylinder, it will produce **one line**, a degenerate conic.



If the plane is parallel to the sides of the cylinder and cuts into the cylinder, it will produce **two parallel lines**.



If the plane is parallel to the sides of the cylinder but doesn't touch the cylinder at all, it will produce **no point** of intersection.

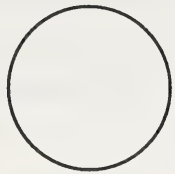
These cases are examples of degenerate conics.

Summary of Conic Shapes

The primary conic sections are the ellipse, the parabola, and the hyperbola.

ellipse parabola hyperbola

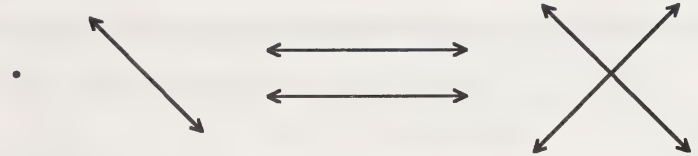
The circle is a special type of ellipse. Therefore, a circle is a limiting case of an ellipse.



circle

Summary of Degenerate Conics

The degenerate conic sections are as follows: the point, the line, two parallel lines, two intersecting lines, and the case of no locus.



point line two parallel lines two intersecting lines

Example 1

Draw a diagram of a double-napped cone with a vertical axis of symmetry and a common vertex, such that the vertices lie between the two cones.

Solution:

Now try the following questions.

Make a list of ways you can produce each of the following shapes:

1. a circle
2. a parabola
3. an ellipse
4. a hyperbola

For solutions to **Activity 1**, turn to **Appendix A, Topic 1**.

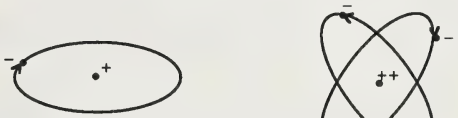
Scientists have discovered some amazing resemblances between the conic sections and astronomy.

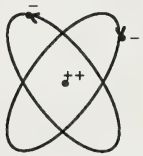
They have discovered planets and other celestial bodies that move about in space following paths that are consistent with those of the conic sections. In other words, some planets or comets will follow the path of a circle or an ellipse. There are bodies that cross the heavens in straight lines. There are others that come into the solar system from deep space, then turn around and head back into deep space in paths that resemble parabolas. The cases go on.

Many of the shapes of the conic sections can be observed in your everyday life.

Elliptical Cases

The paths of electrons moving about in an atom are examples of circles and ellipses. There are certain shapes in nature that are very close to being elliptical in shape. Examples of these are as follows: the eyeball of a mammal, the egg of a reptile, certain types of watermelon, raindrops at certain elevations, and so on.



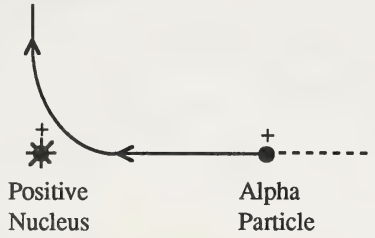


Parabolic Cases

When you play volleyball, you hit the ball upwards. The path it follows up and then down maps out the path of a parabola. When you squirt water up with a garden hose, the path of the water is also a parabola. Another example is the path of a missile. You can think of many more examples.

Hyperbolic Cases

When a positively charged alpha particle is repelled by a positively charged nucleus, the path the particle follows is hyperbolic.



Try the following questions.

5. Get some moist clay or plasticine and mold it into the shape of a right circular cone. Use a sharp blade (knife or razor) to make each of the following cuts. What do you observe for each cut?

   a. vertical cut

   b. horizontal cut

   c. slant cut

6. Cut two cones from styrofoam or any other material. Make sure the two cones are of the same size. Attach the two cones at their vertices so that they form a vertical double cone. Make a vertical cut through the double cone not passing through the vertex. What do you observe at the cross section?

7. Take the clay or plasticine and roll it into a cylinder. Make the following cuts and state what you observe at the intersection of each cut.

   a. Slice it perpendicular to the vertical axis.

   b. Slice it at an angle to the axis that is less than 90° but greater than 0.

8. Get a flashlight and shine a beam of light on a wall in a dark room. What happens to the light on the wall in the following situations?

   a. The beam is perpendicular to the wall.

   b. The beam is held at an oblique angle to the wall.

   c. The flashlight is moved closer to the wall.

   d. The flashlight is moved away from the wall.

9. Pour a coloured solution into a cone-shaped cup.

   a. Hold the cup so the central axis is vertical. What is the shape of the conic formed by the surface of the solution?

   b. Tilt the cup at an angle between 45° and 60°. What is the shape of the conic formed?

10. State the shape you would get if you were to perform the following experiments.

    a. Pour a litre of water into an empty car tire that is sitting upright.

    b. Cut a cylindrical pipe at an angle of 45°, and view the cross section.

    c. Cut the pipe at various angles, and view the cross sections.

11. a. Flatten a cone-shaped cup. Mark and cut a slice not passing through the axis.

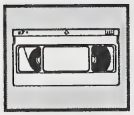
    b. Take another cup, but do not flatten it. Cut off a slice parallel to the generator. What is your observation?

12. What conditions are necessary to produce the conic sections using a cone and an intersecting plane?

13. A circle is the limiting case of the ellipse. Explain.

14. If a circle is drawn on a thin rubbery sheet, how could you transform it into an ellipse?

For solutions to **Activity 1**, turn to **Appendix A, Topic 1**.



If you have access to a videocassette player, you may find it helpful to view the program **Cutting the Curves**. This is program 1 in the series *Discovering Conics*. This video shows how the various conics including the degenerate forms can be produced by cutting a cone and a cylinder with a plane.

Topic 2 Describing Conics Through the Parameters

Introduction

You will discover through open investigation and exploration some basic principles regarding the laws of the general quadratic relation. You will also be studying the effects of the parameters in the following general quadratic relation:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

How do various combinations of these parameters affect the resulting conic? All of the four main conics – the circle, ellipse, parabola, and hyperbola – will be discussed, as well as some of the degenerate cases. You are expected to use graphing calculators and/or appropriate computer software. Notice that the emphasis is on investigation and exploration and not on specific arithmetic calculations.

What Lies Ahead

Throughout this topic you will learn to

1. understand some of the effects of changing the numerical coefficients one at a time in the equation
   $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
2. understand some of the effects of changing A and C in
   $Ax^2 + Cy^2 + Dx + Ey + F = 0$
3. understand some of the effects of changing A or C one at a time in the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$
4. understand the effects of making D or E equal to zero in
   $Ax^2 + Cy^2 + Dx + Ey + F = 0$
5. understand some of the effects of changing D, E, or F to various values in the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$
6. understand some of the effects of changing B in
   $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

Now that you know what to expect, turn the page to begin your study of describing conics through the parameters.

Exploring Topic 2

Activity 1

Understand some of the effects of changing the numerical coefficients in the equation
$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

In order to study the effects of changing the values of A, B, C, D, E, and F in the general quadratic equation, you could change these values one at a time, two at a time, or up to all six at a time while the others are held constant. This would give you
${}_6C_1 + {}_6C_2 + {}_6C_3 + {}_6C_4 + {}_6C_5 + {}_6C_6$ or 63 possible combinations. Changing each of these values or parameters requires trying several values for each. This means you would need to graph at least three or four equations for each change in parameter. This would give you at least 200 to 300 equations to graph.

Until now this was an impossible task. However, with the advent of graphing calculators and computer graphing programs, studying the changes of the coefficients by graphing a large number of equations is fairly quick and easy.

You may use any graphing calculator or graphing program for this topic.

Some suggested graphing programs include *Master Grapher and 3D Grapher* (available for Macintosh, Apple II series, or IBM PC) and *Zap-a-Graph* (available for Macintosh). Some of these programs can be purchased from the Learning Resources Distributing Centre. You can also try any software distribution store. If they do not have a program, they may be able to bring it in for you.

You can enter the equations directly into the calculator by entering the following equations:

- $Y_1=(-(BX+E)+\sqrt{\ }((BX+E)^2-4C(AX^2+DX+F)))/(2C)$
- $Y_2=(-(BX+E)-\sqrt{\ }((BX+E)^2-4C(AX^2+DX+F)))/(2C)$

See your manual on how to graph equations. However, it is easier to use a program which allows you to just enter the values for the parameters.

The following are sample calculator programs for the TI-81 and Casio fx-7700 graphing calculators. If you have a graphing calculator on loan from the Alberta Distance Learning Centre, it should be preprogrammed so you may omit this section on programming the graphing calculators.

The first part is the program which you enter, and the second part is the instructions for how to enter the program. The instructions tell you exactly which keys to press to get a particular symbol. Notice that you may have to press several keys to get one letter or symbol. Follow the instructions carefully when entering the program. Follow the program line by line to be sure you are entering it exactly as given.

Sample Program for the TI-81 Graphing Calculator

Read all of this sample program and the key strokes chart very carefully before you begin programming your calculator. Then return to this point and start entering the program. If at some point while entering the program you leave the calculator on long enough that it automatically shuts off, just press [ON] and your program will return.

To enter the program turn the calculator on, push [PRGM], then [▶] to EDIT. You should now be at Program 1. Press [▼] if you want to select a different program number. Press [ENTER]. Enter the word CONICS here. To enter the word CONICS, press C O N I C S and [ENTER]. (Note: In the program line you do not need [ALPHA] to enter letters. After this first line you will need to press [ALPHA] before each letter you want to enter.)

The standard range setting of $x = -10$ and 10 and $y = -10$ and 10 for a graphing calculator does not give a perfect circle when a circle is graphed since the screen is wider than it is high. To get equal units on each axis, you need to set particular range values. These range values are given in the sample program for the TI-81 on the following page.

Since you are using a program in the calculator, it will be best if the range values are set in the program. This way even if you change the range outside the program, each time you go back to the program the range will reset to the same values.

Enter the range values, starting at the first blank line after you have entered the word CONICS in the PRGM # line. Enter −14.1, which is the minimum value you want for x. Be sure to use the [(-)] for negative. Next press the [STO▶] key which is found in the lower left corner of the keyboard. Next press the [VARS] key. This takes you to a new menu. On this new menu in the right top corner you will find the abbreviation RNG. This stands for range. Press [◀] once. This will move the highlight on to the RNG abbreviation and give you a new menu displaying Xmin, Xmax, and so on.

Press [1] on the keyboard since you want to store −14.1 for Xmin. Then press [ENTER] to complete the instruction. Next enter the value 14.1. Again press [STO▶], then [VARS], and then [◀] to highlight RNG. Now press [2] on your keyboard to store the value 14.1 as Xmax. Again press [ENTER] to complete this line of instruction. Continue this sequence until you have entered all seven values into the program. When you have finished, your

screen should have the first seven lines of your program on it. Now you are ready to enter the next line of the program.

To enter Disp "A", press PRGM, ▶ to I/O, and use ▲ or ▼ to select Disp. Since Disp is the first line of the menu, you may not have to move up or down. Press ENTER, then press ALPHA and + for the quotation marks, then ALPHA and A for the letter A, and then ALPHA and + for one more set of quotation marks. Press ENTER to move to the next line. Be sure to press the keys firmly.

When entering the equations, be sure to use the (-) key for the negative sign and the − key for minus.

Now enter the remainder of the program using the keystrokes listed following the program. Notice most of the symbols require more than one keystroke. You may edit the program by using the arrow keys to move up or down or left or right. Use DEL to delete an entry. Use INS to insert a missing entry.

```
Prgm1: CONICS
-14.1→Xmin
14.1→Xmax
1→X sc1
-9.3→Ymin
9.3→Ymax
1→Y sc1
1→res
:Disp "A"
:Input A
:Disp "B"
:Input B
:Disp "C"
:Input C
:Disp "D"
:Input D
:Disp "E"
:Input E
:Disp "F"
:Input F
:If C≠0
:Goto 3
:"(-1*A*X^2–D*X–F)/(B*X+E)"→Y1
:DispGraph
:End
:Lbl 3
:"(-1*(B*X+E)+√((B*X+E)^2–4*C*(A*X^2+D*X+F)))/
(2*C)"→Y1
:"(-1*(B*X+E)–√((B*X+E)^2–4*C*(A*X^2+D*X+F)))/
(2*C)"→Y2
:DispGraph
:End
```

Symbol	Keystrokes for TI-81
"	[ALPHA], [+]
A	[ALPHA], [MATH]
B	[ALPHA], [MATRX]
Disp	[PRGM], [▶] to I/O, [▲] or [▼] to highlight Disp, [ENTER]
Input	[PRGM], [▶] to I/O, [▲] or [▼] to highlight Input, [ENTER]
≠	[2nd], [MATH], arrow to ≠, [ENTER]
Y_1	[2nd], [VARS], arrow to Y_1, [ENTER]
*	[×]
exponent 2	[x^2]
$\sqrt{}$	[2nd], [x^2]

Symbol	Keystrokes for TI-81
/	[÷]
→	[STO▶]
If	[PRGM], arrow to If, [ENTER]
Goto	[PRGM], arrow to Goto, [ENTER]
DispGraph	[PRGM], [▶] to I/O, [▲] or [▼] to highlight DispGraph, [ENTER]
End	[PRGM], arrow to End, [ENTER]
Lbl	[PRGM], [ENTER]

When you have finished entering the program, push [2nd] and then [CLEAR]. To run the program push [PRGM], [▼] to choose the correct program, then [ENTER], and then [ENTER] again. Now you can enter your values of A, B, C, D, E, and F. Press a number for A, then [ENTER], press a number for B, then [ENTER], and so on until you have entered all six values. If you have negative numbers for A, B, C, D, E, or F, then you must use the [(-)] key.

Sample Program for the Casio fx-7700G Graphing Calculator

Read all of this sample program and the keystroke chart very carefully before you begin programming your caculator. Then return to this point and start entering the program. If at some point while entering the program you leave the calculator on long enough that it automatically shuts off, just press [ON] and return to your program by pressing [MODE], [2], arrow down to your program number, and [EXE].

To begin programming turn the calculator on, press [MODE], then [2], then arrow down [▼] to an empty program number, and then [EXE].

First you must enter the range. The standard range settings of $x = -10, 10$ and $y = -10, 10$ will not give a perfect circle on the calculator since the scales are not the same on both axes. To set the same scales on both axes you need to set the following range into the program. Press the range key and enter the numbers -14.1, 14.1, 1, -9.3, 9.3, and 1, and then press [EXE]. It should look like this:

Range $-14.1, 14.1, 1, -9.3, 9.3, 1$

Be sure to put the commas between the numbers. The comma is [SHIFT] and the arrow key [→].

Note: In the Casio fx-7700G graphing calculator you may use the subtraction key [−] for both the negative symbol and when you need to subtract. However, do not use the negative key [(-)] for subtraction.

Now enter the following program using the keystrokes listed following the program. To get letters use [ALPHA] followed by the letter. For example, the letter C is [ALPHA], [ln]. You may edit the program by using the arrow keys to move up or down or right or left, then using [DEL] to delete an entry and INS (use [SHIFT], [DEL]) to add a missing entry. You can also overwrite an entry by doing a new entry while on the entry you want to replace.

CONICS EXE
"A="?→A:"B="?→B:"C="?→C EXE
"D="?→D:"E="?→E:"F="?→F EXE
C=0⇒Graph Y= $(-1\times A\times X^2-D\times X-F)\div(B\times X+E)$▲
Graph Y=$(-1\times(B\times X+E)+\sqrt{\ }((B\times X+E)^2-4\times C\times(A\times X^2+D\times X+F)))\div(2\times C)$ EXE
Graph Y=$(-1\times(B\times X+E)-\sqrt{\ }((B\times X+E)^2-4\times C\times(A\times X^2+D\times X+F)))\div(2\times C)$▲

When programming is completed press [MODE], [1], and [AC] to get out of the programming mode.

To run this program press [SHIFT], [RANGE], [F3], the program number, and [EXE]. To clear the screen press [SHIFT], [F5], and [EXE].

Now you can enter your values of *A*, *B*, *C*, *D*, *E*, and *F*. Press a number for *A* then [EXE]. Press a number for *B*, then [EXE], and so on, until you have entered all six values.

Symbol	Keystrokes for Casio fx-7700G
C	[ALPHA], [ln]
"	[ALPHA], [F2]
=	[SHIFT], [RANGE], [F2], [F1]
?	[SHIFT], [RANGE], [F4]
:	[SHIFT], [RANGE], [F6]
⇒	[SHIFT], [RANGE], [F1], [F1]
Graph Y=	[GRAPH]
exponent 2	[SHIFT], [$\sqrt{\ }$]
▲	[SHIFT], [RANGE], [F5]
EXE, ÷, $\sqrt{\ }$, →, (,), ×	Actual Keys

Now you are ready to try graphing some equations. Follow the instructions at the end of each sample program carefully. You may have to refer to your manual for help on how to set the range and use the zoom features of the calculator. If you are using a particular graphing program, carefully follow the instructions given in the program.

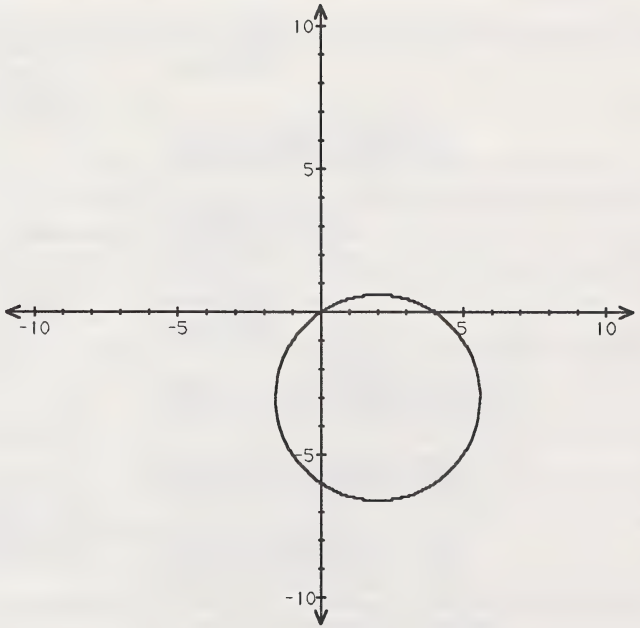
In this section you will be given a number of equations to graph. The resulting graph is given immediately after each equation. Graph each equation with whatever graphing technology you are using and compare it to the given graph. The purpose of this section is to help you familiarize yourself with using graphing technology and to give you confidence that you are obtaining the correct graphs. If you are already familiar with the graphing calculator or computer program you are using, you may omit this section and go on to the first investigation.

Example 1

Graph the following equations and compare your graphs with the ones that are given.

- $x^2 + y^2 - 4x + 6y = 0$

Solution:

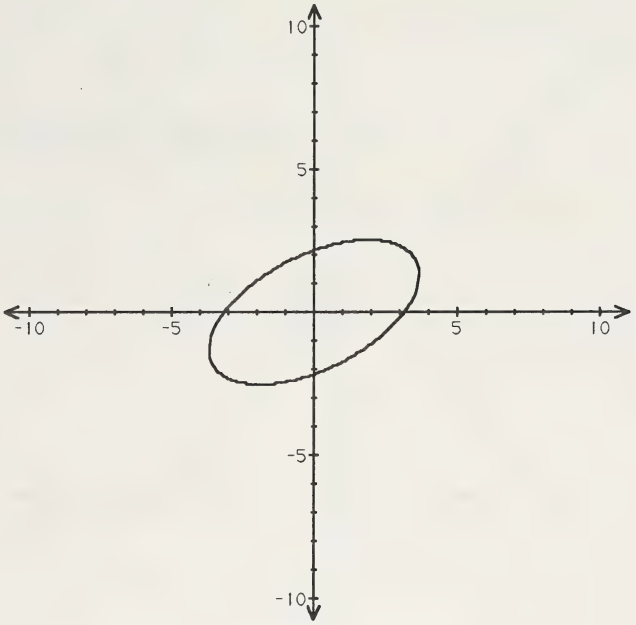


Did you get a graph that looks like the one shown? If you are using a graphing calculator, you will have a break on each side of the circle. This is due to the way the graphing calculator makes the graph. The graphing calculator makes the graph by graphing the upper half and lower half separately, resulting in a split graph.

If your graph does not resemble the given graph, try entering the values again. Be sure you are entering a zero value for each missing parameter. In this case B and F are zero.

- $8x^2 - 12xy + 17y^2 - 80 = 0$

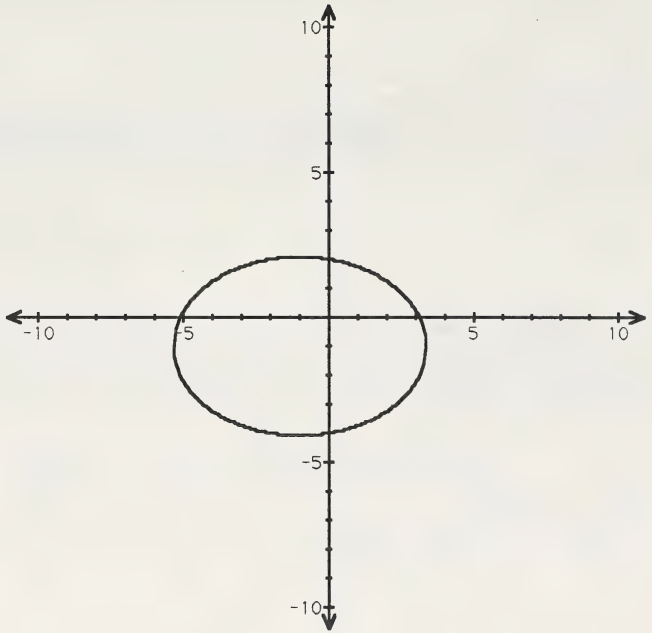
Solution:



Compare your graph to the given graph.

- $x^2 + 2y^2 + 2x + 4y - 16 = 0$

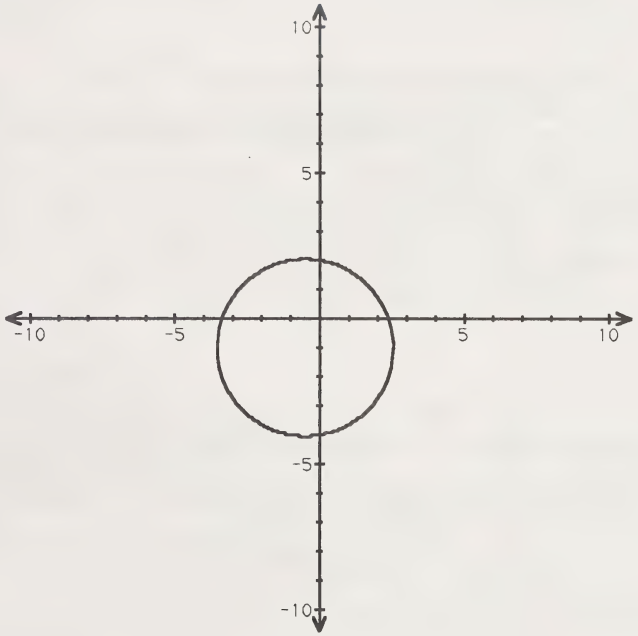
Solution:



Compare your graph to the given graph.

- $2x^2 + 2y^2 + 2x + 4y - 16 = 0$

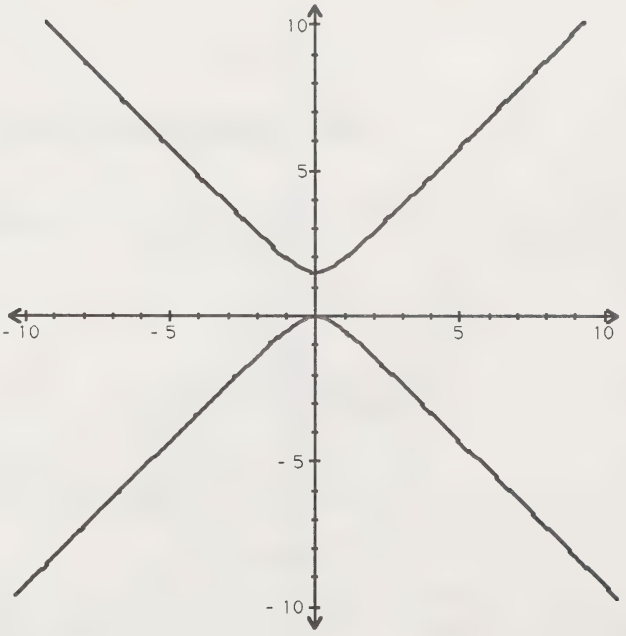
Solution:



Compare your graph to the given graph.

- $2x^2 - 2y^2 + 3y = 0$

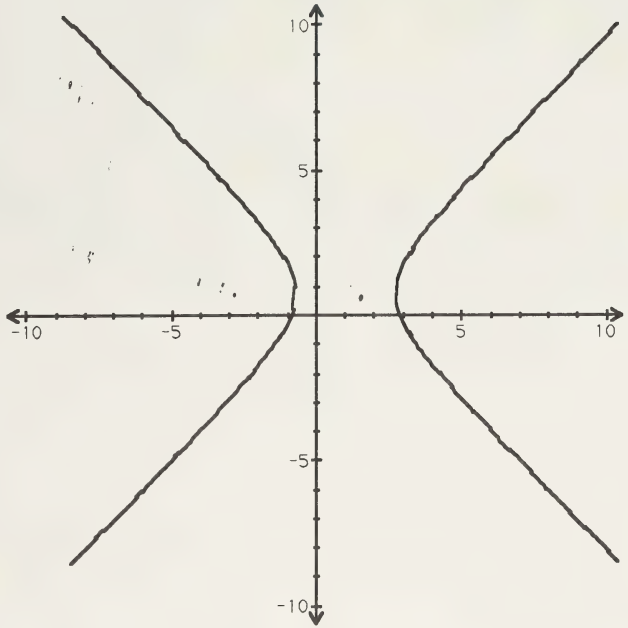
Solution:



Compare your graph to the given graph.

- $2x^2 - 2y^2 - 4x + 3y - 5 = 0$

Solution:



Compare your graph to the given graph.

Now that you have tried graphing various equations, you should be ready to use the graphing calculator or computer program to study the effects that changing the parameters has on the graph. You will discover the effects that each variable produces on the graphs through investigations.

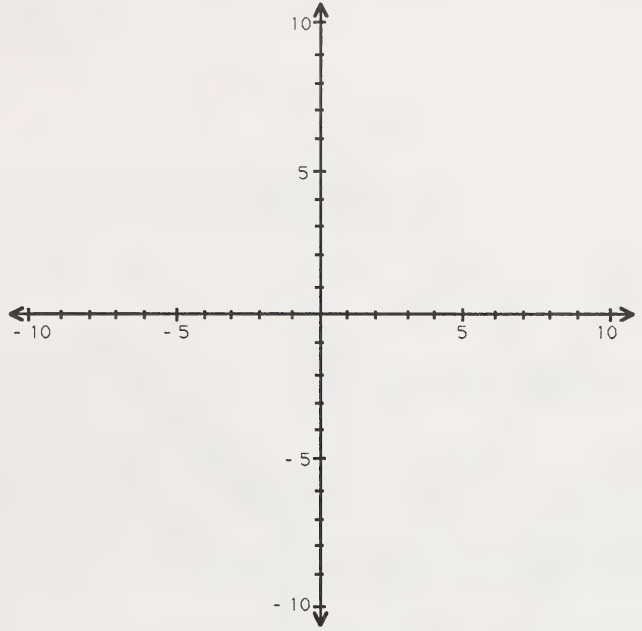
If you are using a graphing calculator, you must draw the resulting graph, give the equation, and write any observations or conjectures on the grids and in the space following each equation. Complete each investigation first, and then turn to **Appendix A** for the correct solutions.

Investigation 1

Graph the following relations using a graphing calculator or computer program. Be sure to draw your graphs on the grids provided so you can use them for future reference. Below each grid be sure to answer the following questions for each equation.

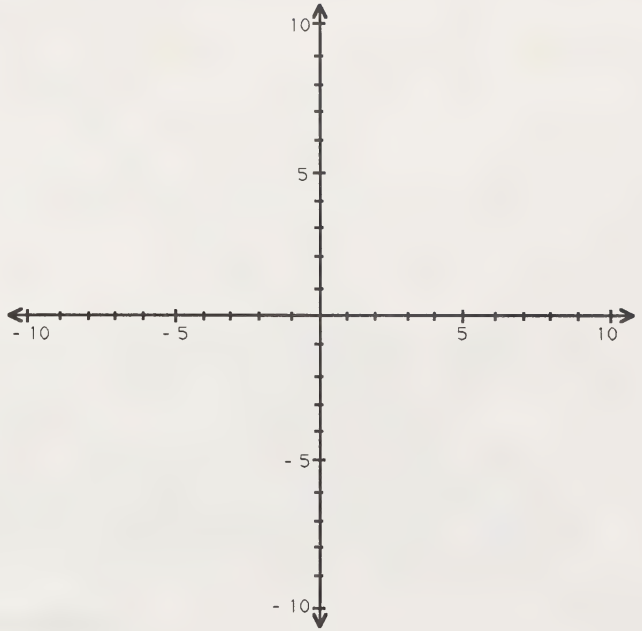
- What type of curve is created by each relation?
- What are the values of the coefficients for each of the curves?

a. $x^2+y^2+4x+8y-10=0$



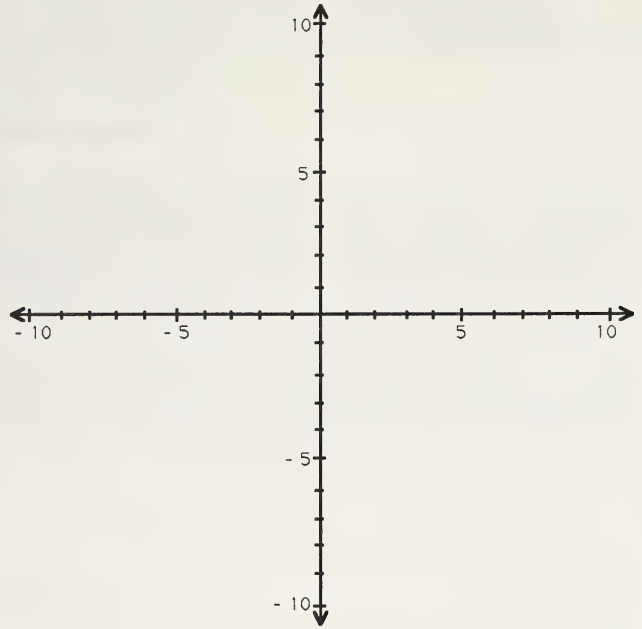
Observations:

b. $4x^2+y^2+4x+8y-3=0$



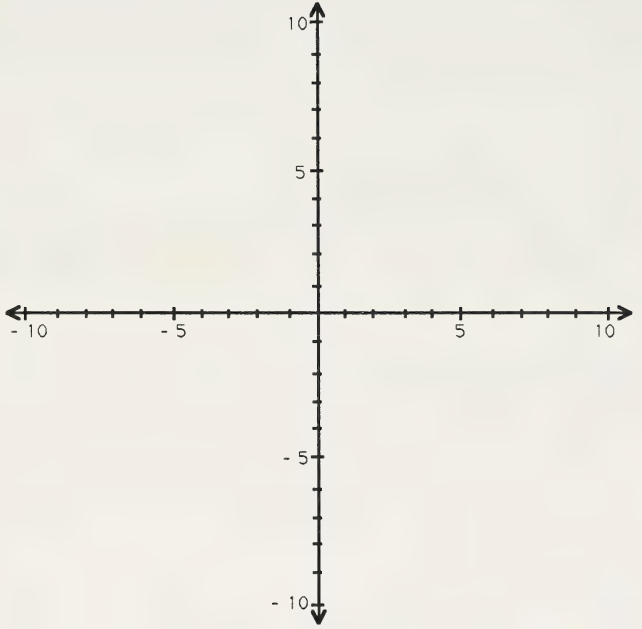
Observations:

c. $x^2 - y^2 + 4x - 8y - 16 = 0$



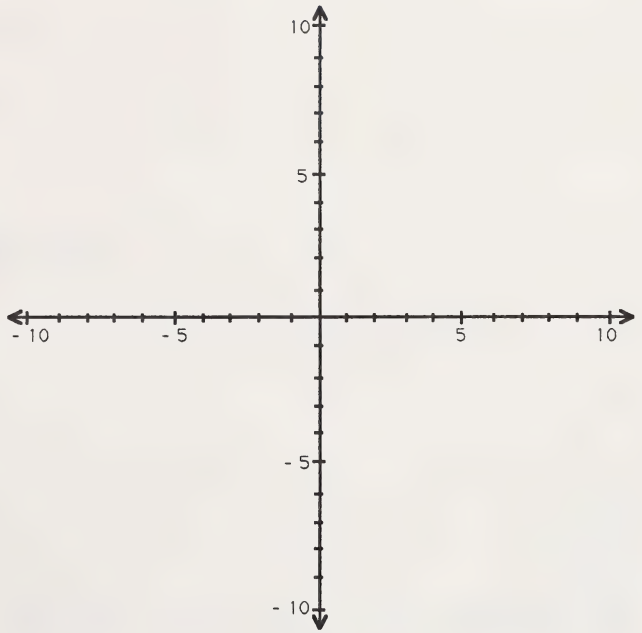
Observations:

d. $x^2 + 4x + 6y - 1 = 0$



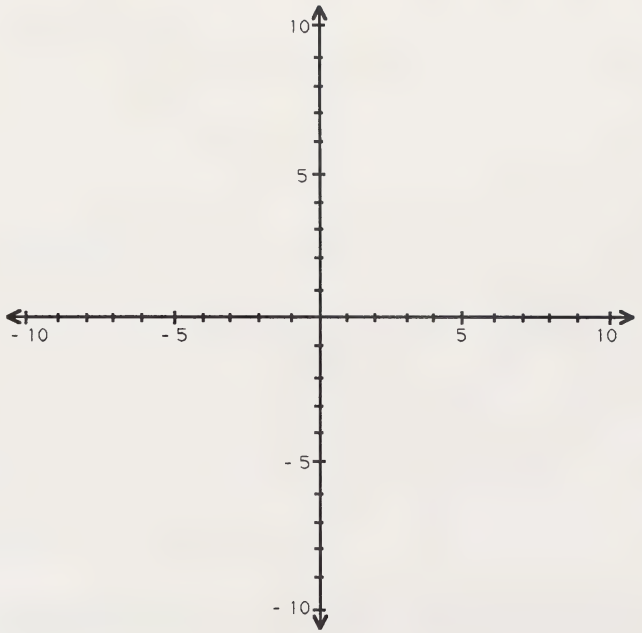
Observations:

e. $x^2 + xy + y^2 + 2x + 2y - 16 = 0$



Observations:

f. $2x^2 + xy + 2x - 25 = 0$



Observations:

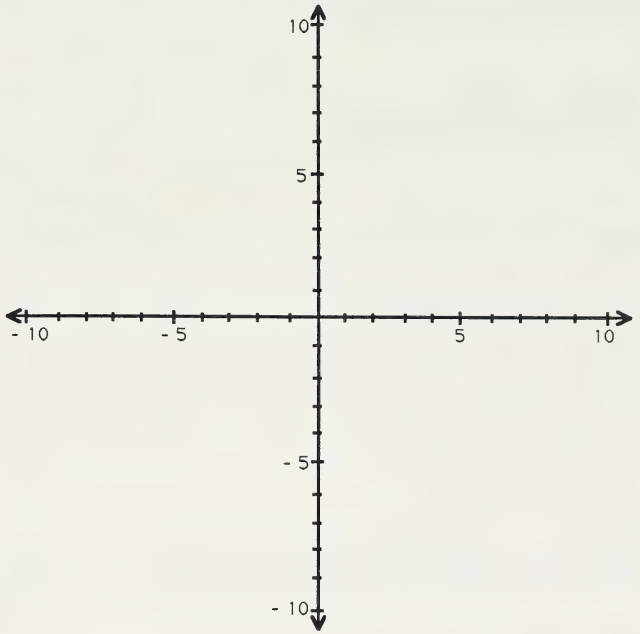
In the first topic you looked at the conics through the intersection of a plane and a cone. You came across four distinct conics that could be created: the circle, the ellipse, the parabola, and the hyperbola. In the six equations in this investigation you have found these four curves as well.

Investigation 2

In this investigation an equation from the previous investigation is selected and the coefficients A, B, C, D, E, and F are systematically changed. After each question keep a log of the changes for each coefficient. You may wish to use a different color pencil for each graph. Since you are putting several graphs on one set of axes, try to place the x- and y-intercepts as accurately as possible. If the curves are the same type (for example, both are ellipses), pay close attention to any differences. Summarize the effect of changing each coefficient in the space provided below each question.

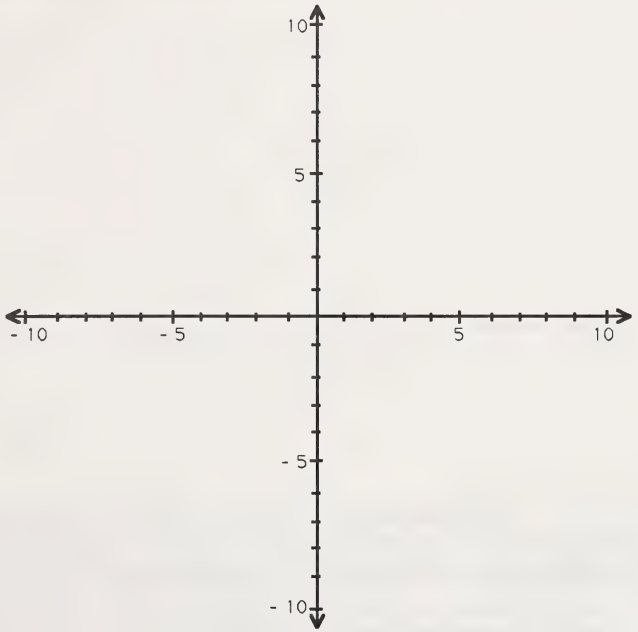
Equation: $x^2 + xy + y^2 + 2x + 2y - 16 = 0$

a. Change A using $A = -10, -2, 0, 2$, and 9.



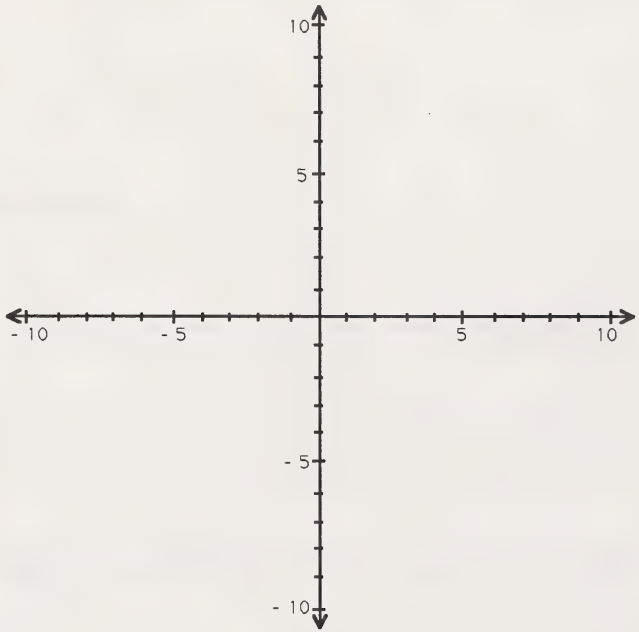
Observations:

b. Change B using $B = -10, -2, 0, 2,$ and 9.



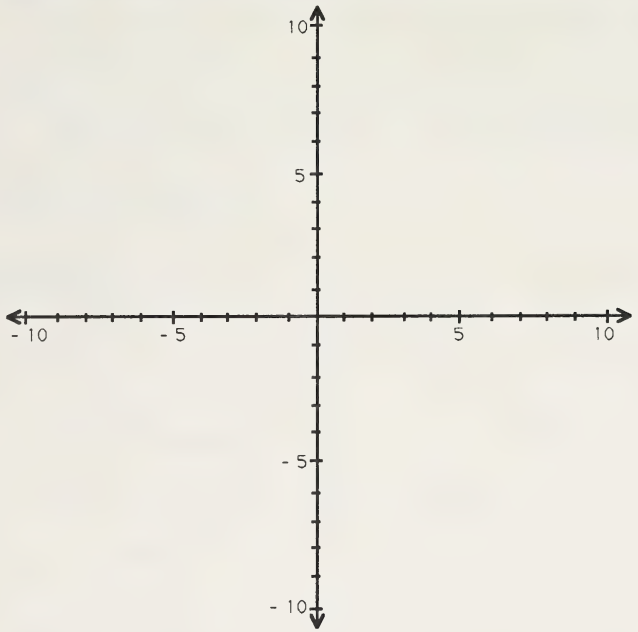
Observations:

c. Change C using $C = -10, -2, 0, 2,$ and 9.



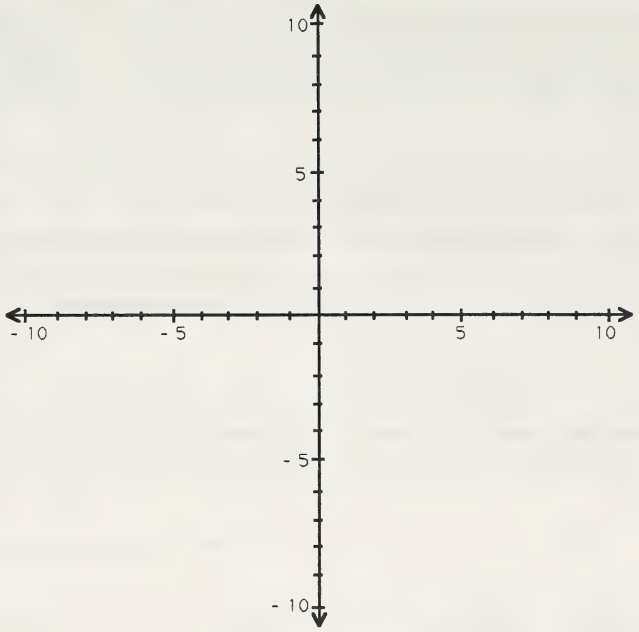
Observations:

4. Change D using $D = -10, -2, 0, 2,$ and 9.



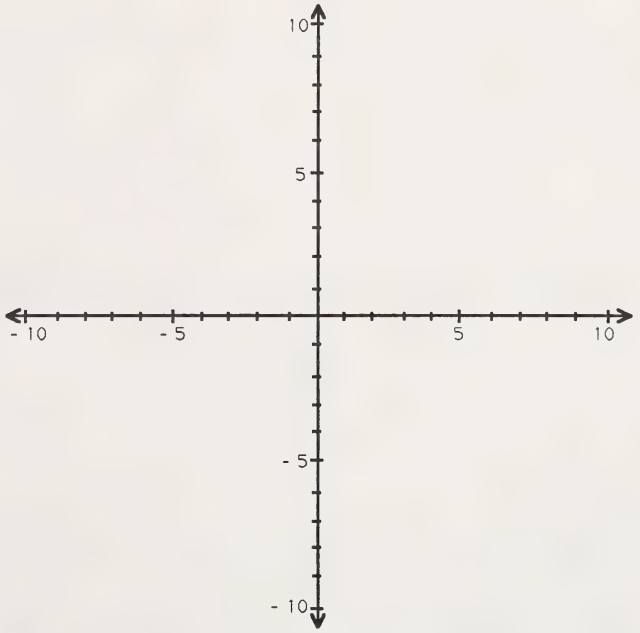
Observations:

e. Change E using $E = -10, -2, 0, 2,$ and 9.



Observations:

f. Change F using $F = -10, -2, 0, 2$, and 9.



Observations:

This concludes your Activity 1 investigations. Now summarize some of the effects on the curves obtained when each of the various parameters is changed.

1. What kinds of changes did you see when each parameter was changed?

   a. when A was changed

   _____

   _____

   _____

   b. when B was changed

   _____

   _____

   _____

   c. when C was changed

   _____

   _____

   _____

d. when D was changed

e. when E was changed

f. when F was changed

Notice the effects of changing A, B, and C are quite complex. In particular the effects of B can be very involved, and unexpected things can happen.

For solutions to **Activity 1**, turn to **Appendix A, Topic 2.**

Activity 2

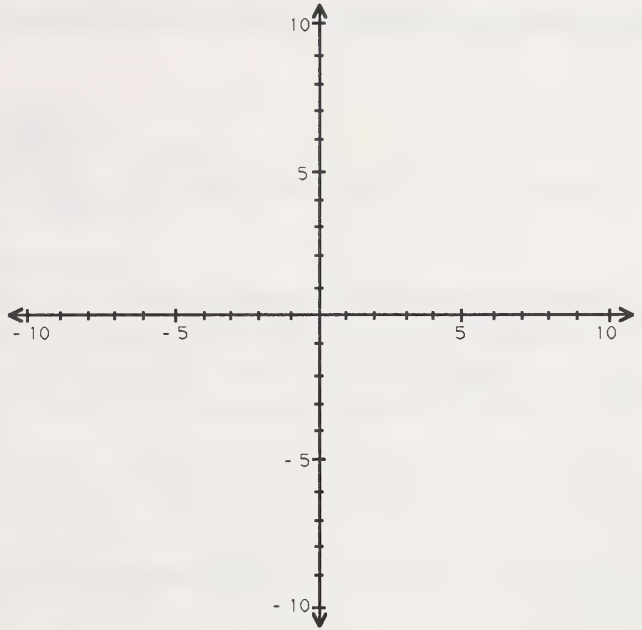
Understand some of the effects of changing A and C in $Ax^2 + Cy^2 + Dx + Ey + F = 0$.

Now you will begin your investigations of the specific changes that occur in the general conic when $B = 0$. The general form of the equation which you will be using is $Ax^2 + Cy^2 + Dx + Ey + F = 0$. What shapes are possible when A and C are varied with $B = 0$. To simplify the observations, D and E will be made equal to zero where possible.

You will be required to pick values for one or two of the parameters. Keep the values you pick to within positive or negative 10, unless you are specifically asked to use very large values or very small values.

Investigation 3

Investigate the shape of the curve when $A = C$ and $B = 0$. Use the equation $Ax^2 + Cy^2 - 16 = 0$. Use various positive and negative values for A and C. Try some large values, say 100 or 1000, as well. Graph your results on the grid provided, and answer the questions which follow.



Observations:

a. What is the resulting figure?

_____

_____

b. What happens when A and C are large?

_____

_____

c. What happens when A and C are negative?

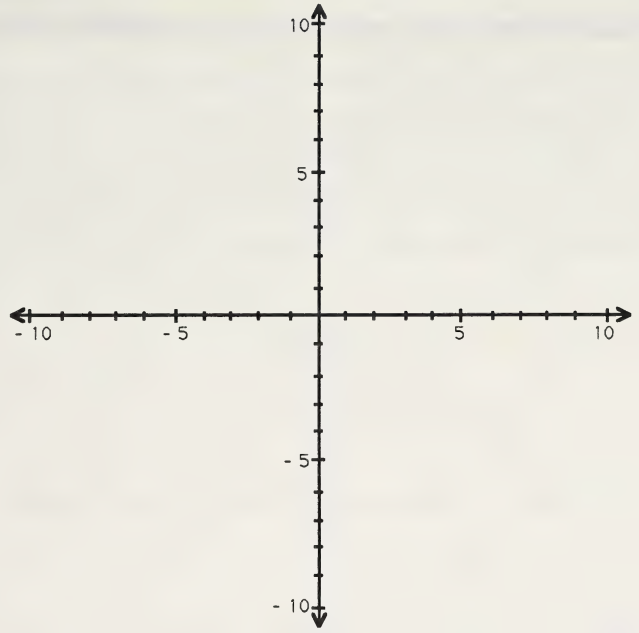
_____

_____

Investigation 4

What is the shape of the curve when either $A = 0$ or $C = 0$? Use the equation $Ax^2 + Cy^2 + 4x + 8y - 16 = 0$. Answer the questions which follow.

Try some positive and negative values for A when $C = 0$. Draw your graphs on the grid provided. Next try some positive and negative values for C when $A = 0$. Draw the resulting graphs on the grid.



Observations:

a. What is the shape of the graph when either $A = 0$ or $C = 0$?

_____

b. If A is a positive value and $C = 0$, which way does the parabola open?

_____

c. If A is a negative value and $C = 0$, which way does the parabola open?

_____

d. If $A = 0$ and C is a positive value, which way does the parabola open?

_____

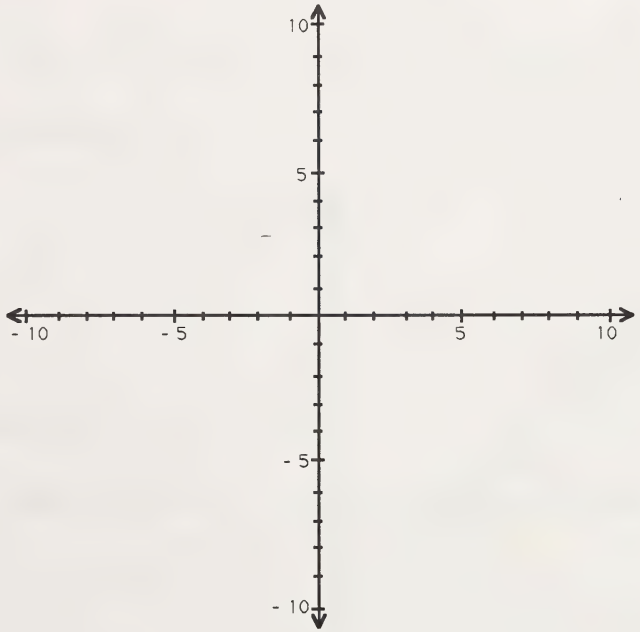
e. If $A = 0$ and C is a negative value, which way does the parabola open?

_____

You should recall from previous mathematics studies that there were similar effects when you varied the sign of a in $ax^2 + bx + c = y$.

Investigation 5

Investigate the shape of the curve when A and C are not equal and $B = 0$. Use the equation $Ax^2 + Cy^2 - 16 = 0$. Use both positive and negative values for A and C. Also, vary the size of A as compared to C. Graph your results in the grid provided, and answer the questions which follow.



Observations:

a. What are the resulting shapes when A and C have the same sign?

_____

_____

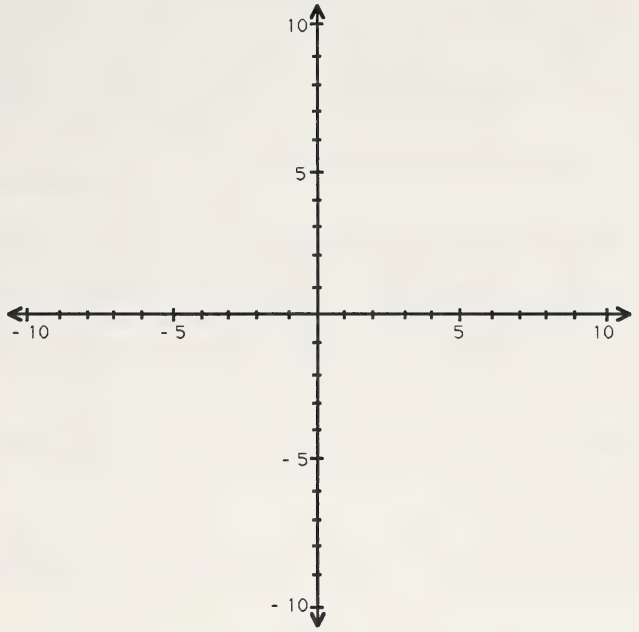
b. What are the resulting shapes when A and C have different signs?

_____

_____

Investigation 6

Investigate the effects on the ellipse when A and C are interchanged. Using $Ax^2 + Cy^2 - 16 = 0$, pick values for A and C where A is less than C. Draw the ellipse on the grid provided. Now interchange the values of A and C. Draw this ellipse on the same grid. What can you conclude about the relative size of A and C and the orientation of the graph? Space is provided for your conclusion following the grid.

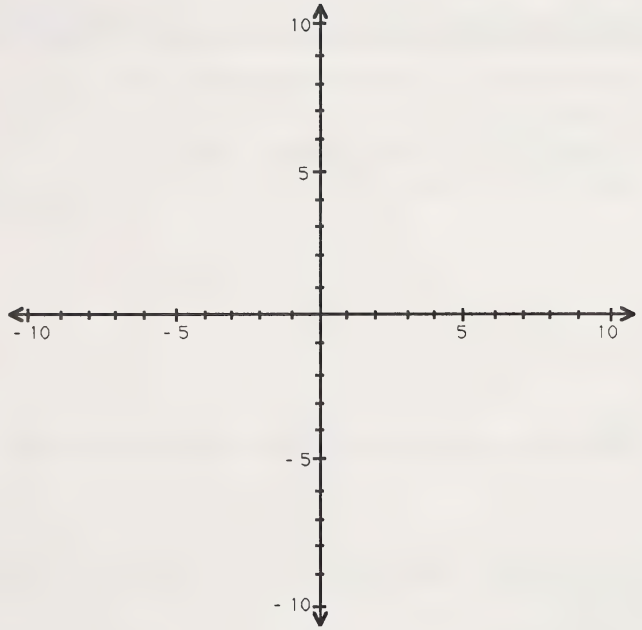


Observations:

Conclusion:

Investigation 7

Investigate the effects of interchanging A and C on the hyperbola. Use $Ax^2 + Cy^2 - 16 = 0$. Pick values for A and C that will give a hyperbola. Draw the resulting graph. Now interchange A and C. Notice this involves a sign change. Draw the resulting graph. Try just exchanging the signs. Draw the resulting graph. What can you conclude about the sign of A and C and the position of the graph? Space is provided for your conclusion following the grid.



Observations:

Conclusion:

This concludes your Activity 2 investigations. Now referring to Investigations 3 through 7 summarize the types of shapes that are possible if B is set equal to zero and A and C are varied. Explore the variations when A and C are equal, when either A or C is equal to zero, when A and C are not equal but have the same signs, and when A and C are not equal and have different signs.

Summary:

For solutions to **Activity 2**, turn to **Appendix A, Topic 2**.

Activity 3

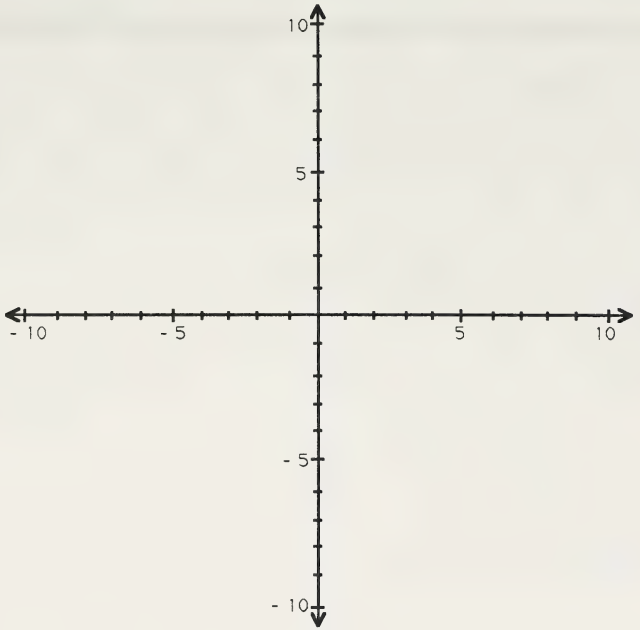
Understand some of the effects of changing A or C one at a time in the equation

$Ax^2 + Cy^2 + Dx + Ey + F = 0.$

In the previous activity you saw that if you varied A and C, you could get one of the four basic conics – a circle, an ellipse, a parabola or a hyperbola – depending on the values for A and C. In Investigation 3 you saw that changing the size of A and C gave larger or smaller circles. In this activity you will vary A or C one at a time and see how the different values affect the ellipse, the parabola, and the hyperbola.

Investigation 8

In this investigation you will study the effect of changing the value of A in the equation of a parabola, $Ax^2 + 4x - 2y - 8 = 0$. Try some positive and some negative values for A. Record your results on the grid provided, and answer the questions which follow.



Observations:

a. How does the direction the graph opens change as the value of A goes from positive to negative?

_____

b. How does the shape of the graph change as the value of A becomes larger?

_____

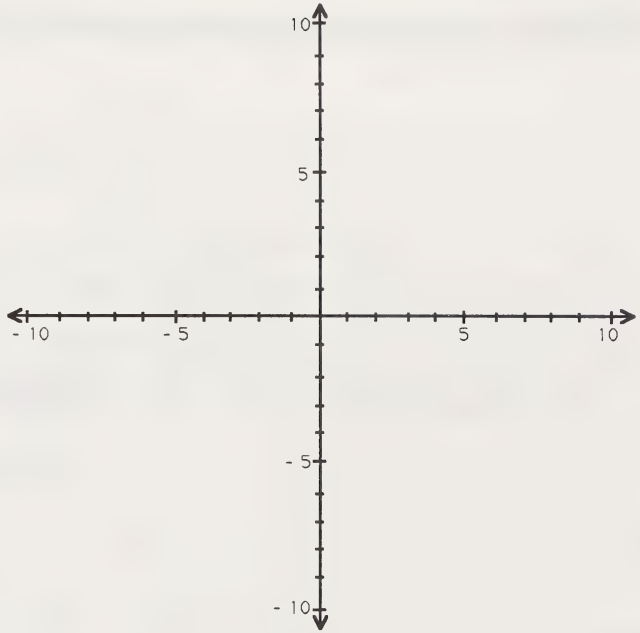
c. Do you think the effects will be similar if you let $A = 0$ and change the value of C? If you are not sure, try it.

_____

Notice that the graph opens down when A is negative and up when A is positive. This is opposite to what you saw in Investigation 4. This is because the sign of E is negative in this equation while E was positive in the equation in Investigation 4. This will be studied further when you look at the effects of D and E.

Investigation 9

Using $Ax^2 + 7y^2 - 36 = 0$ vary the size of A. Use $A = 1, 3$, and 5. Draw each graph on the grid provided. Label the value of A for each curve. Answer the questions which follow.



Observations:

a. What happens to the shape of the ellipse as the value of A is changed from 1 to 3 to 5? (Note: Here, A is less than C.)

_____

_____

b. What can you conclude about the shape of the ellipse when the value of A gets closer to the value of C?

_____

_____

c. What value would you use for A to make this ellipse into a circle?

_____

d. What do you think the effect will be if you make $A = 7$ and vary the value of C. If you are unsure, test the values of 1, 3, and 5 for C. (Note: Here, C is less than A.)

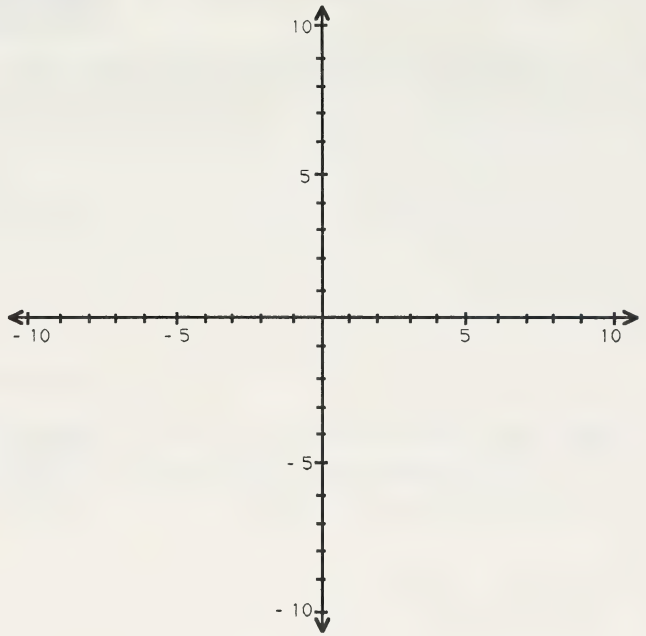
_____

_____

Notice that both the horizontal and vertical ellipses approach a circle as the values of A and C approach the same value. When the values of A and C are the same, you have a circle. You can say that an ellipse changes into a circle when the values of A and C become equal.

Investigation 10

Using $Ax^2 - 4y^2 - 16 = 0$, vary the size of A to investigate the effects of changing the size of A on the hyperbola. Use $A = 1, 4,$ and 6. Draw the graphs on the grid provided. Be sure to label the value of A on each curve. Answer the questions which follow.



Observations:

a. What happens to the shape of the hyperbola as the value of A is changed from 1 to 4 to 6?

b. What can you conclude about the size of A regarding the shape of a hyperbola?

c. Is the effect the same if you let $A=-4$ and use the values of 1, 4, and 6 for C? If you are unsure, test the values of 1, 4, and 6 for C in the equation $-4x^2+Cy^2-16=0$.

Summarize the effects of changing A or C when you have the equations of a parabola, ellipse, and hyperbola. Refer to Investigations 8 through 10 to help you complete your summary.

Summary:

For solutions to **Activity 3**, turn to **Appendix A, Topic 2.**

Activity 4

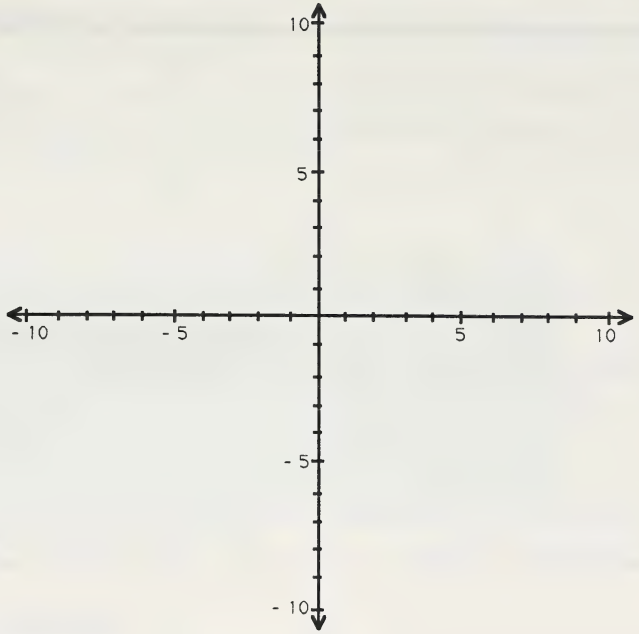
Understand some of the effects of making D or E equal to zero in $Ax^2 + Cy^2 + Dx + Ey + F = 0$.

In this activity you will try to find the values that put the vertex of a parabola or the centres of a circle, an ellipse, and a hyperbola at the origin of the coordinate plane.

Investigation 11

First you will find the values that put the vertex of a parabola at the origin. As you already know from Investigation 4, a parabola has either A or C equal to zero.

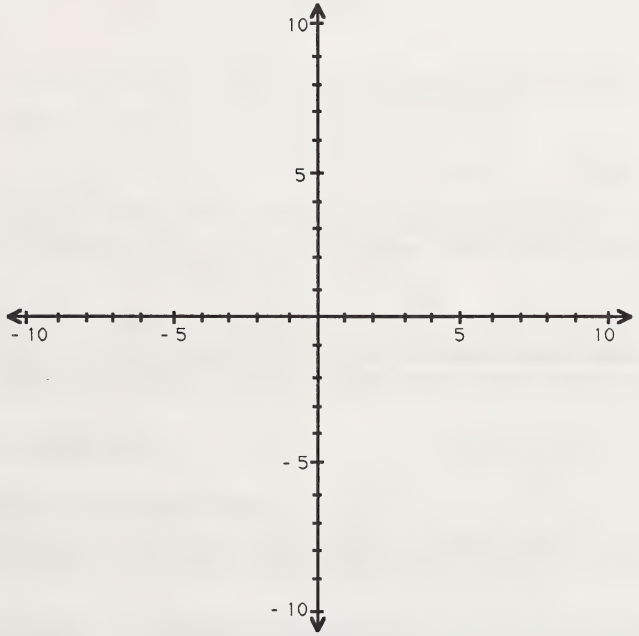
Use the equation $7x^2 + Dx + 3y = 0$. Choose various values for D. Draw your graphs on the grid provided, and answer the question which follows.



Observations:

a. What value of D puts the vertex of the parabola at the origin?

Use the equation $5y^2 + 3x + Ey = 0$. Choose various values for E. Draw the graphs on the grid provided, and answer the question which follows.



Observations:

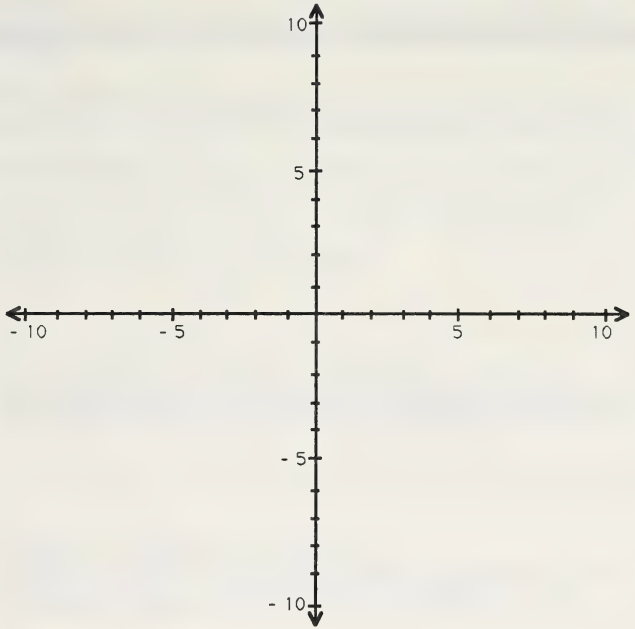
b. What value of E puts the vertex of the parabola at the origin?

Investigation 12

Now you will find the values that put the centres of a circle, ellipse, and hyperbola at the origin.

Circle

Graph the equation $3x^2 + 3y^2 + Dx + Ey - 81 = 0$ on the grid provided. Let $D = E$ and choose various values for D and E. Then, answer the question which follows.



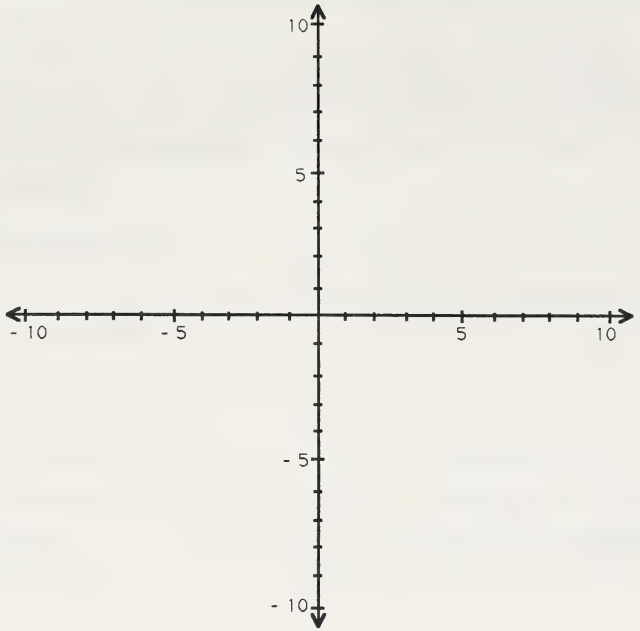
Observations:

a. What values for D and E put the centre of a circle at the origin?

_____

Ellipse

Graph the equation $2x^2 + 7y^2 + Dx + Ey - 49 = 0$ on the grid provided. Let $D = E$ and choose various values for D and E. Then, answer the questions which follow.



Observations:

b. What values for D and E put the centre of an ellipse at the origin?

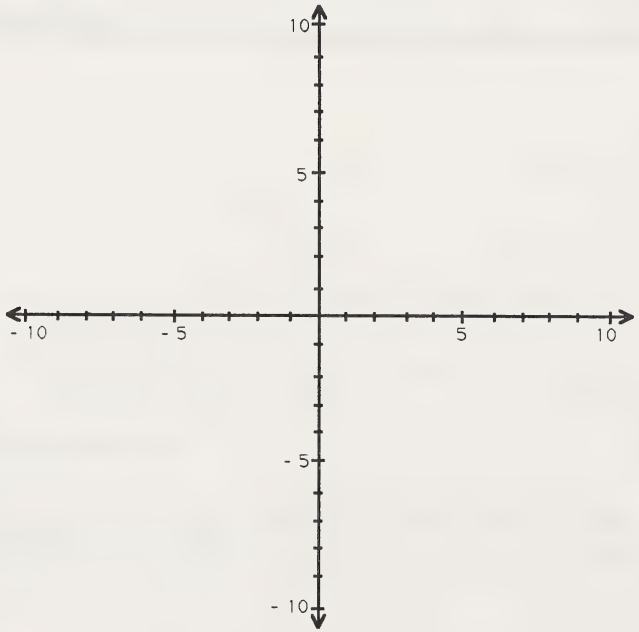
_____

c. Exchange the values of A and C. Let $D = E$ and try various values for D and E. Do the same values of D and E put the centre of the ellipse at the origin?

_____

d. Try other values for A and C which will make an ellipse. Do the same values for D and E put the ellipse at the origin?

_____



Observations:

Hyperbola

Graph the equation $2x^2 - 9y^2 + Dx + Ey - 25 = 0$ on the grid provided. Let $D = E$ and choose various values for D and E. Then, answer the questions which follow.

e. What values of D and E put the centre of the hyperbola at the origin?

_____

f. Exchange the values of A and C. Let $D = E$ and try various values for D and E. Do the same values of D and E put the centre of the hyperbola at the origin?

_____

g. Try other values for A and C which will make a hyperbola. Do the same values for D and E put the hyperbola at the origin?

_____

This concludes your Activity 4 investigations. In summary, what values of D and E put all of the curves at the origin?

Summary:

_____

_____

_____

For solutions to **Activity 4**, turn to **Appendix A, Topic 2.**

Activity 5

Understand some of the effects of changing D, E, or F to various values in the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$.

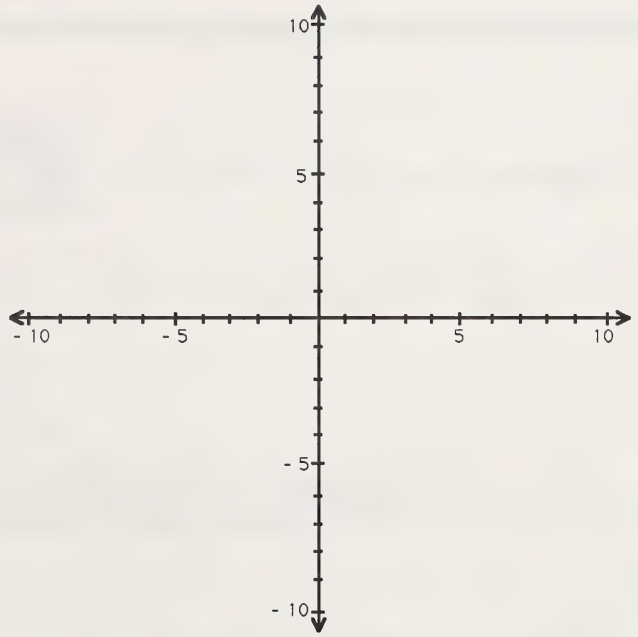
In Activity 4 you saw that if you made D and E equal to zero, you could produce a circle, ellipse, parabola, or hyperbola at the origin. In this activity you will look at the effects of D and E when they are not equal to zero. Answer the questions included within each investigation.

Investigation 13

In this investigation you will look at the effects of changing D on each of the curves.

Circle

Start with the equation $4x^2 + 4y^2 - 81 = 0$. Graph this circle on the grid provided. Now try $D = 9$ and $D = -9$ and graph the results on the same grid. Record each value for D beside the corresponding graph.



Observations:

a. What is the effect when $D = 9$?

_____

b. What is the effect when $D = -9$?

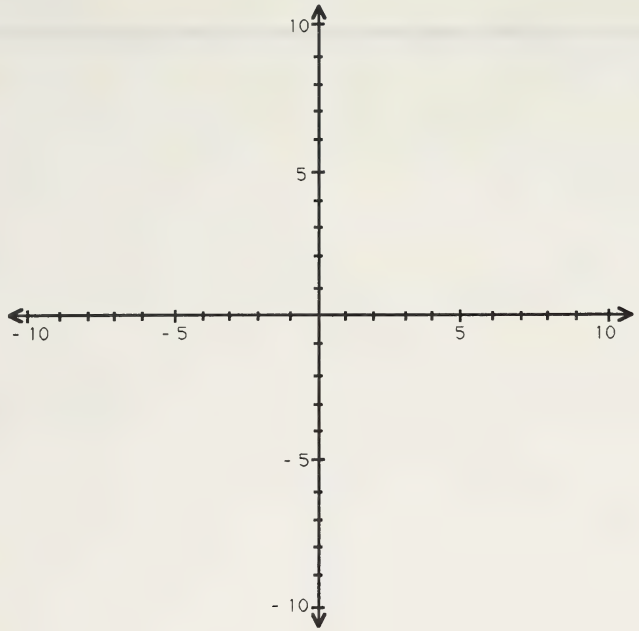
_____

c. Try other positive and negative values for D. Is the effect the same?

_____

Ellipse

Start with the equation $x^2 + 4y^2 - 16 = 0$. You should recognize this as an ellipse centred at the origin. Draw its graph on the grid provided. Try $D = 5$ and $D = -5$. Draw the graphs for each on the grid. Be sure to record each value for D beside the proper graph.



Observations:

d. Which way does the graph move when D is positive?

_____

e. Which way does the graph move when D is negative?

_____

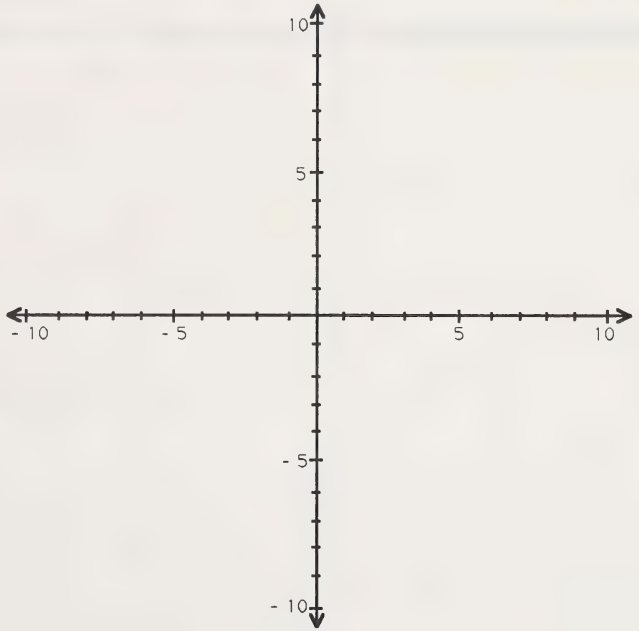
f. Interchange A and C in the equation. Try $D=5$ and $D=-5$ again. Is the effect the same?

_____

Notice that in addition to moving left or right, the ellipse increases in size. This is because, although changing the sign of D moves the graph only left or right, changing the size of D also affects F. As you will see in Investigation 15, changing F changes the size of the ellipse. However, the important thing to remember is that the dominant effect of changing D is to move the graph left or right.

Parabola

Start with the equation $3x^2+Dx+9y=0$. When $D=0$ you know the graph has its vertex at the origin. Try $D=7$ and $D=-7$ and graph the results on the following grid.



Observations:

g. How does the position of the parabola change when $D = 7$ as compared to the position when $D = 0$?

_____

(Did you notice that the graph moved up as well as left? This is because changing the size of D affects F as well.)

h. What is the effect on the position of the graph when $D = -7$?

_____

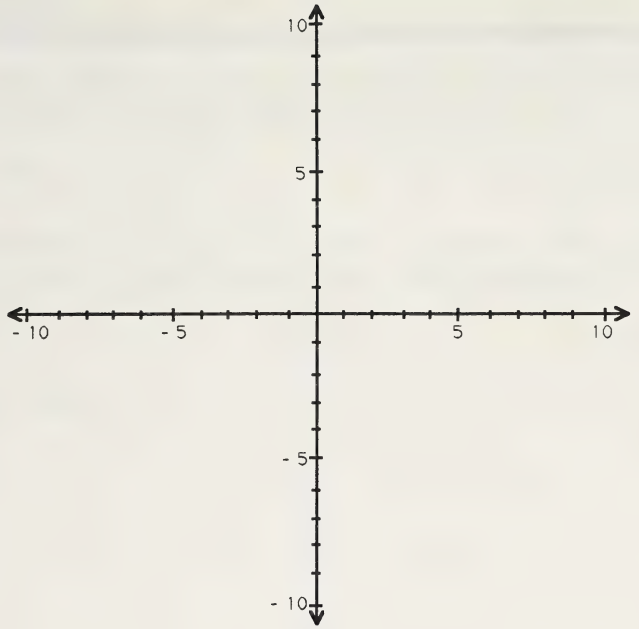
(Notice with a sign change the movement is only left or right. Hence, for a change in D the dominant effect is for the graph to move left or right.)

i. Try other positive and negative values for D. Is the effect the same?

_____

Hyperbola

Start with the equation $x^2 - 4y^2 - 16 = 0$. Graph this equation on the grid provided. This is a hyperbola on the x-axis centred at the origin. Now try $D = 5$ and $D = -5$. Draw these graphs on the following grid.



j. What is the effect when $D = 5$?

_____

k. What is the effect when $D = -5$?

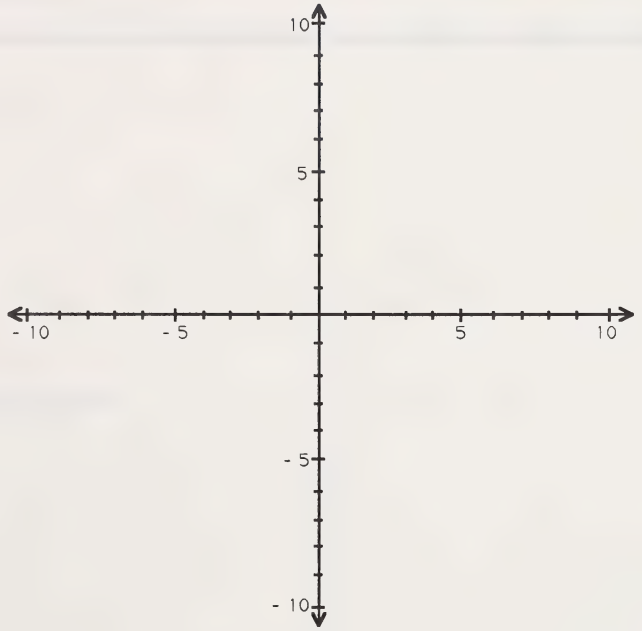
_____

l. Try other positive and negative values for D. Is the effect the same?

_____

Now try the equation $-x^2 + 4y^2 - 16 = 0$. This is a hyperbola on the y-axis. Graph this equation on the following grid. Now try $D = 5$ and $D = -5$ and draw the graphs on the same grid.

Observations:



Observations:

m. What is the effect when $D = 5$?

n. What is the effect when $D = -5$?

o. Are the changes here the same as the changes that occurred with the hyperbola on the x-axis?

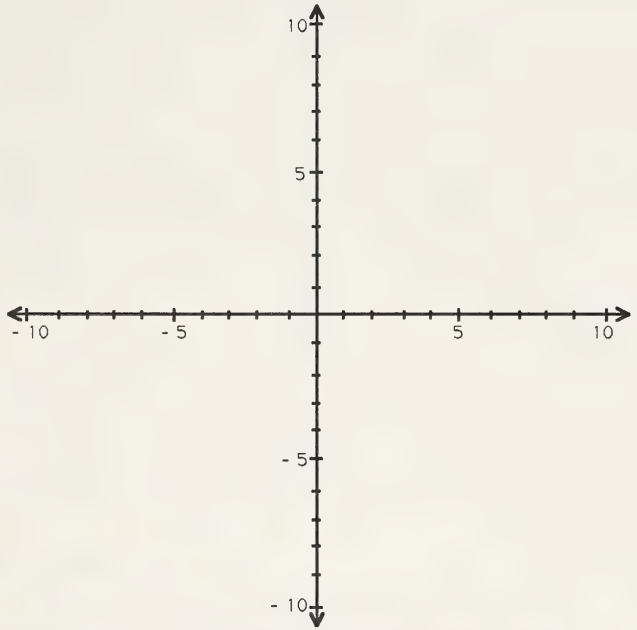
p. Summarize the movements that occur with each type of curve when D is positive and when D is negative.

Investigation 14

In this investigation you will look at the effects of changing E on the graphs of the circle, ellipse, parabola, and hyperbola.

Circle

Start with the equation $3x^2 + 3y^2 - 49 = 0$. Draw the graph on the grid provided. Now try $E = 8$ and $E = -8$. Graph the results on the grid.



Observations:

a. What is the effect when $E = 8$?

_____

b. What is the effect when $E = -8$?

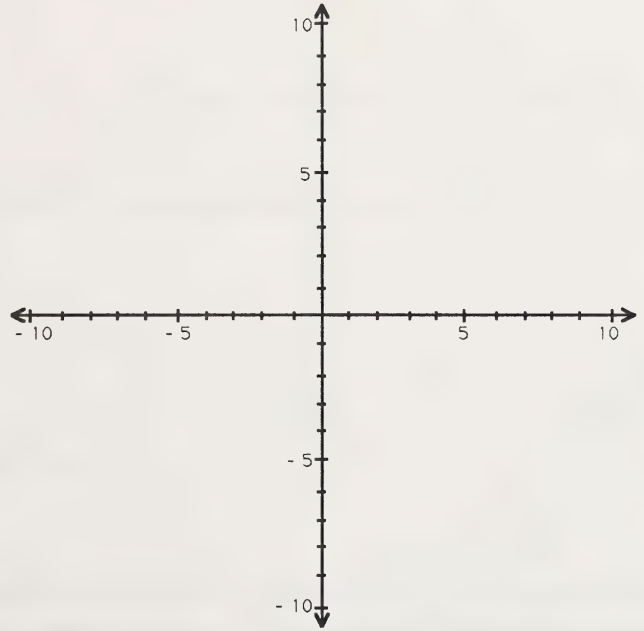
_____

c. Try other positive and negative values for E. Is the effect the same?

_____

Ellipse

For this investigation start with the equation $x^2 + 4y^2 - 16 = 0$. Now try $E = 5$ and $E = -5$. Draw these three graphs on the following grid.



Observations:

d. Which way does the graph move when E is positive?

_____

e. Which way does the graph move when E is negative?

_____

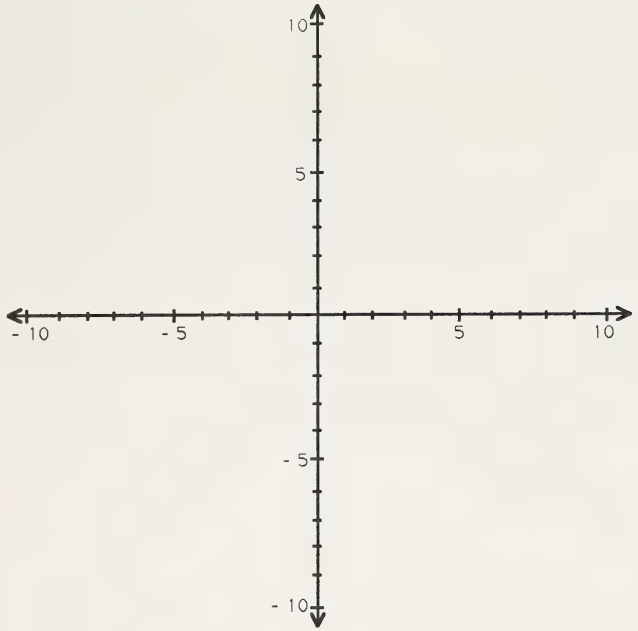
f. Interchange A and C in the equation. Try $E = 5$ and $E = -5$ again. Is the effect the same?

_____

Note again that although changing the size of E increases the size of the ellipse, the dominant effect is to move the ellipse up or down.

Parabola

Start with the equation $y^2 + 6x + Ey = 0$. When $E = 0$ you know the graph has its vertex at the origin. Try $E = 4$ and $E = -4$. Graph the results on the grid provided.



Observations:

g. How does the position of the parabola change when $E = 4$ as compared to the position when $E = 0$?

(The graph moves right as well as up because changing the size of E affects F just as changing D affected F in the previous investigation.)

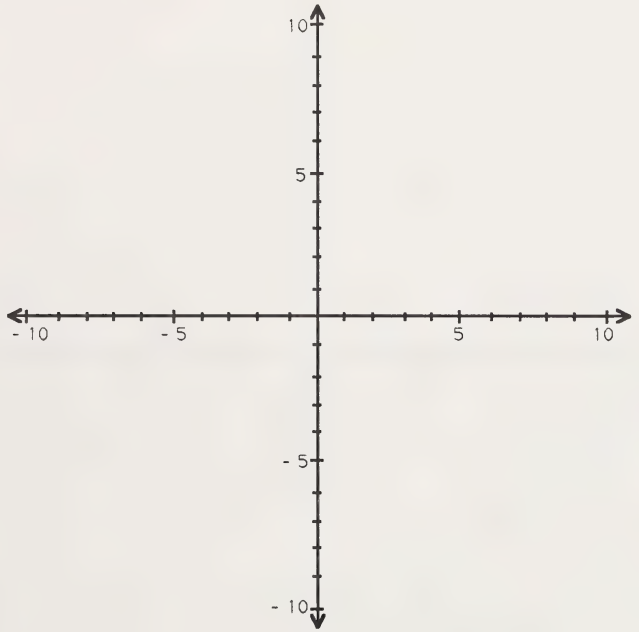
h. How is the position of the graph affected when $E = -4$?

(Notice with a sign change the graph moves up and down only. Hence, for a change in E the dominant effect is up or down.)

i. Try other positive and negative values for E. Is the effect the same?

Hyperbola

Start with the equation $x^2 + 4y^2 - 16 = 0$. Graph this equation on the grid provided. This is a hyperbola on the x-axis. Try $E = 10$ and $E = -10$. Draw the graphs on the grid.



Observations:

j. What is the effect on the graph when $E = 10$?

_____

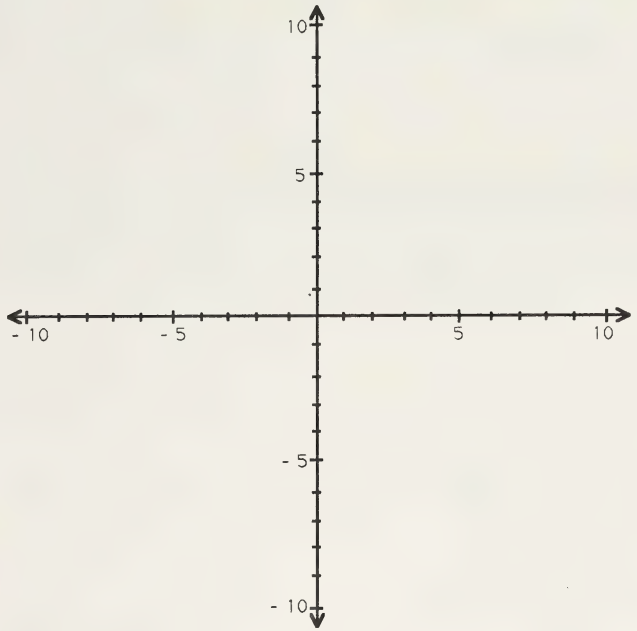
k. What is the effect on the graph when $E = -10$?

_____

l. Try other positive and negative values for E. Is the effect the same?

_____

Now try the equation $-x^2+4y^2-16=0$. Graph this equation on the following grid. This is a hyperbola on the y-axis. Now try $E=10$ and $E=-10$. Draw the graphs on the grid.



Observations:

m. What is the effect on the graph when $E=10$?

n. What is the effect on the graph when $E=-10$?

o. Try other positive and negative values for E. Is the effect the same?

p. How does the movement of a hyperbola on the y-axis compare with the movement of a hyperbola on the x-axis when the value of E is changed?

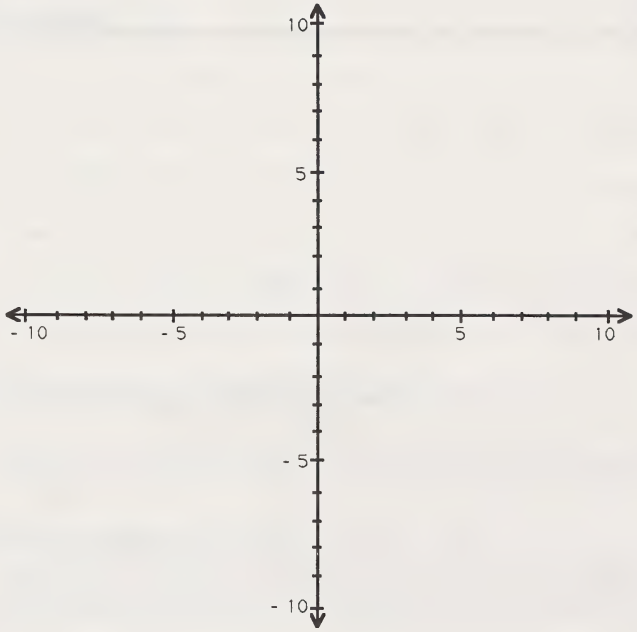
q. Summarize the effects of E on the various curves.

Investigation 15

In this investigation you will look at the effects of changing F on the graphs of the circle, ellipse, parabola, and hyperbola.

Circle

Start with the equation $2x^2 + 2y^2 - 25 = 0$. Draw the graph on the following grid. Then try $F = -36, -49$, and -81. Draw these graphs on the grid.



Observations:

a. What happens to the graph as the absolute value of F becomes larger?

_____

b. Try $F=-1$ and -0.5. Do these values of F give a graph?

_____

(Notice the circle is getting smaller and smaller. If you try negative values less than 0.5, the resolution of the calculator will not allow a graph to be displayed. However, there is still a graph. You could change the range and get a graph which can be seen, but the radius of the circle is still getting smaller.)

c. Try $F=64$. What happens?

_____

d. Why did you get no graph in question c? (Hint: Rewrite the equation as $2x^2+2y^2=-64$.)

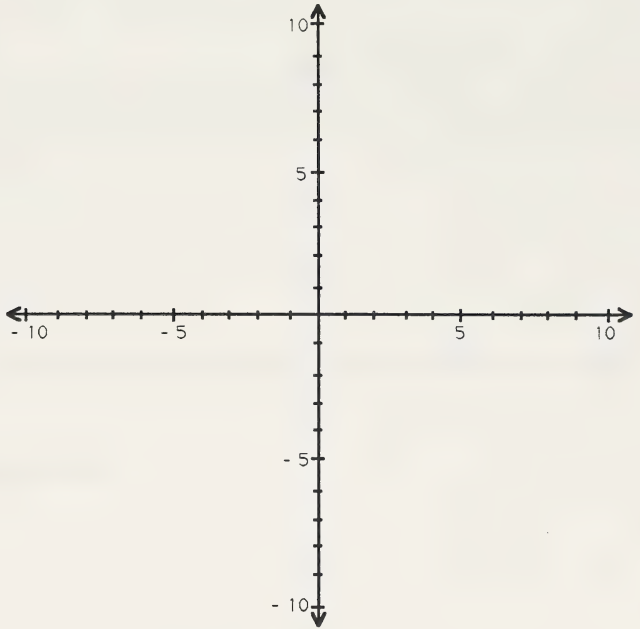
_____

_____

_____

Ellipse

In this investigation start with the equation $x^2+4y^2-4=0$. Try $F=-9, -25,$ and -49. Graph your results on the following grid and answer the questions which follow.



Observations:

e. What happens to the graph when F has various negative values?

_____

f. Interchange A and C. Try $F=-9, -25$, and -49 again. Is the effect the same?

_____

g. Try graphing the equation $x^2+4y^2+16=0$. Do you get any graph? Try $F=1, 25$, and 49. Do you get any graph? Why or why not?

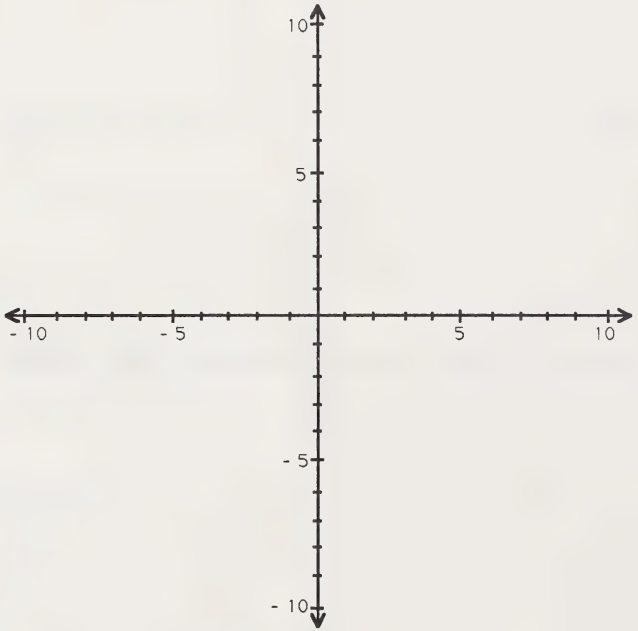
_____

_____

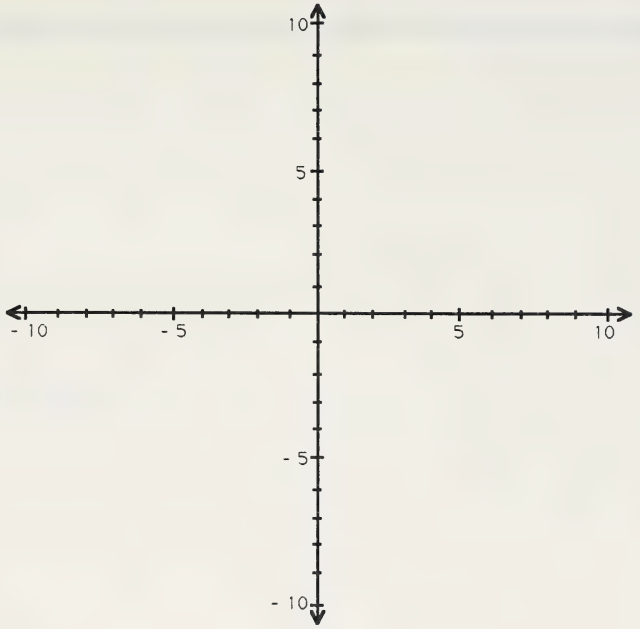
_____

Parabola

Start with the equation $x^2+6y=0$. Draw the graph on the following grid. Try $F=9$ and $F=-9$. Graph the results on the grid.



Observations:



h. What happens to the position of the graph when the value of F is changed from 0 to 9?

_____

i. What happens to the position of the graph when the value of F is changed to -9?

_____

Observations:

j. Try other positive and negative values for F. Is the effect the same?

_____

Now start with the equation $y^2 + 6x = 0$. Try $F = 7$ and $F = -7$. Draw the resulting graphs on the grid and answer the questions which follow.

k. What happens to the position of the graph when F is changed from 0 to 7?

_____

l. What happens to the position of the graph when F is changed to -7?

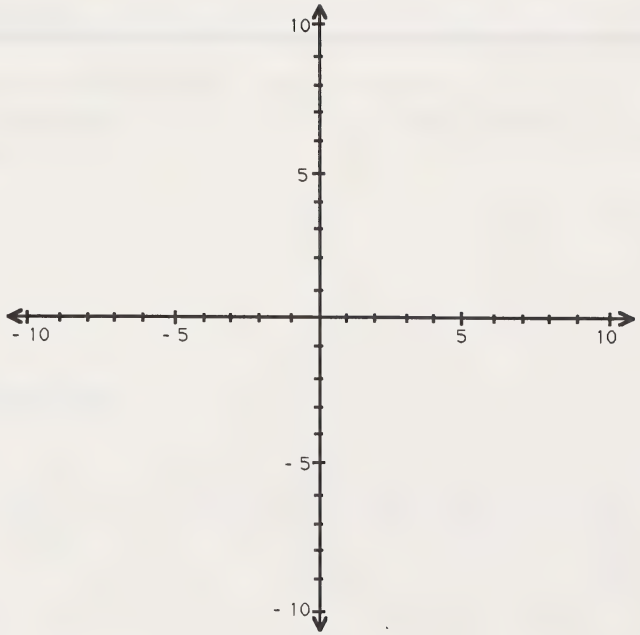
_____

m. Try other positive and negative values for F. Is the effect the same?

_____

Hyperbola

Start with the equation $x^2 - 4y^2 - 1 = 0$. The graph is a hyperbola on the x-axis. Try $F = -16, -36$, and -64. Graph the results on the following grid.



Observations:

n. What is the effect on the graph as F is changed from -16 to -36 to -64?

_____

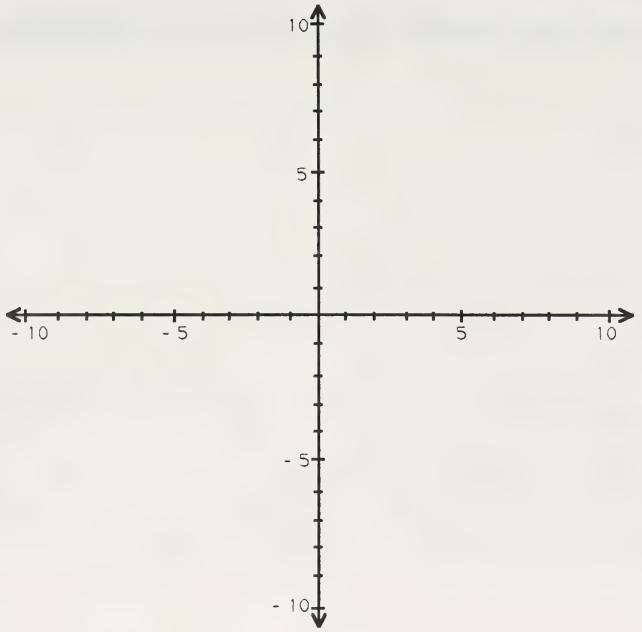
o. Now try $F = 16$. What is the effect on the graph? Why? (Hint: Try multiplying the equation by -1.)

_____

p. Try $F = 49$ and $F = 81$. What is the effect of giving F larger positive values?

_____

Now start with the equation $-4x^2 + y^2 - 1 = 0$. This is a hyperbola on the y-axis. Try $F = -16, -36$, and -64. Graph the results on the following grid.



Observations:

q. What is the effect on the graph as F is changed from -16 to -36 to -64?

_____

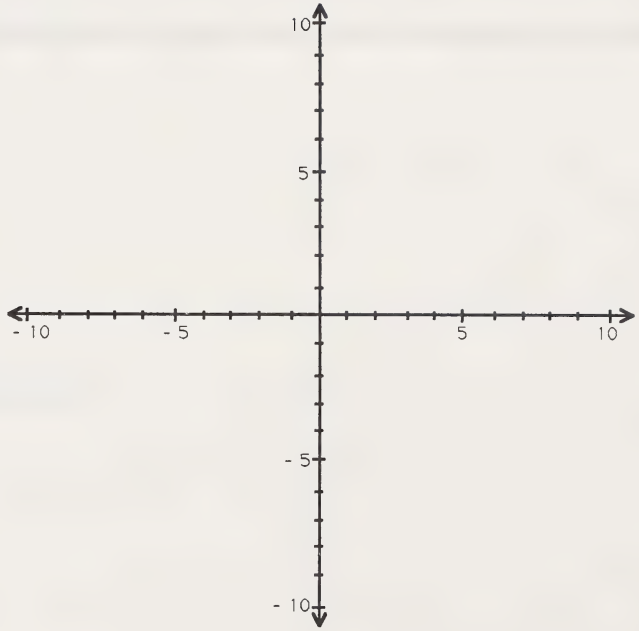
r. Try $F = 16$. What is the effect on the graph? Why?

_____

s. Try $F = 49$ and $F = 81$. What is the effect on the graph?

_____

Using the equation $x^2 - 9y^2 + F = 0$, try $F = 16, 0$, and -16. Graph the results on the following grid and answer the questions which follow.



Observations:

t. What is the effect as F goes from 16 to -16?

u. What happens when $F = 0$? Can you explain why this happens algebraically?

(This is another example of a degenerate conic. In this case you have two intersecting lines.)

v. How does this relate to the cutting of a double cone by a plane as discussed in Topic 1?

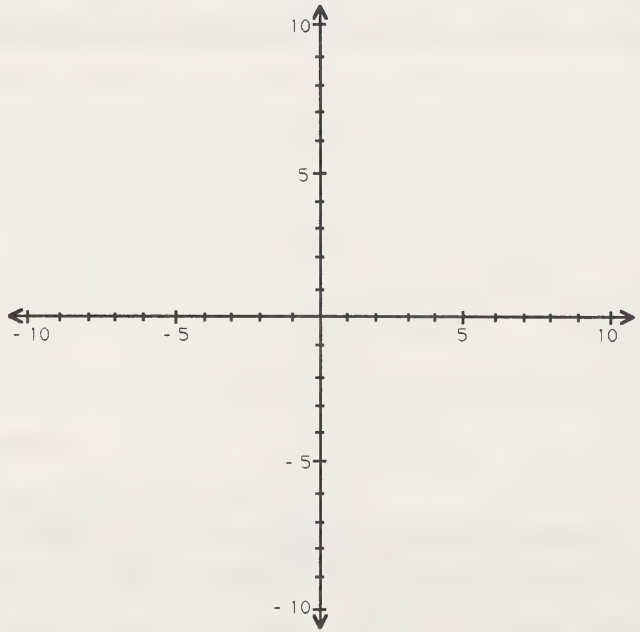
w. Summarize the effects on the circle, ellipse, parabola, and hyperbola when F is changed.

This ends your investigations on the effects on the various curves when D, E, and F are changed. Although there have been a number of different changes, in particular with the F value, you can make some general conclusions about the effects of changing D, E, and F. Summarize the effects of changing D, E, and F with regards to the various curves. Limit your conclusions to a general direction, for example left and right or up and down.

Summary:

Investigation 16

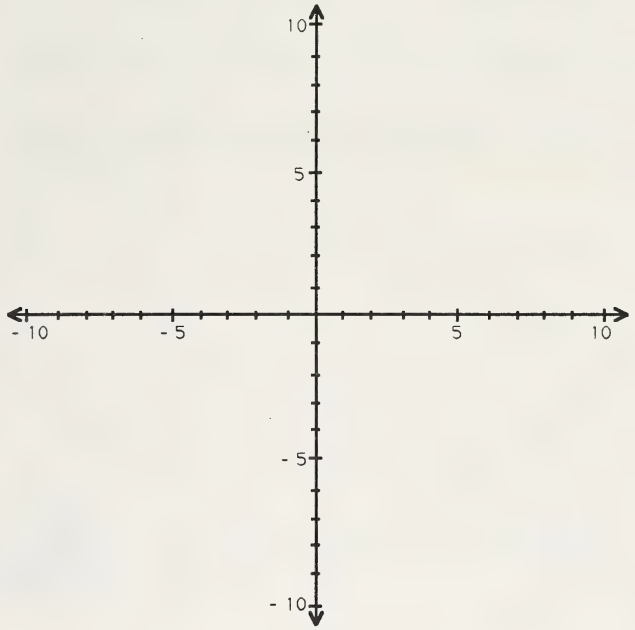
In this investigation define your own equation of an ellipse with centre $(0, 0)$. Then change the values for D and E to define an equation which when graphed has its centre in quadrant 3. Show the graphs and equations for both cases.



Observations:

Investigation 17

Starting with an ellipse with its centre at the origin, make three other ellipses which are successively larger. Give the equations of each.

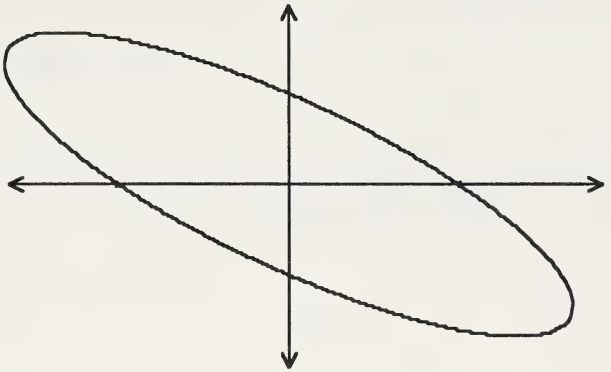


Observations:

Investigation 18

In this investigation you are given various graphs and asked to give a possible combination of coefficients which would determine the kind of curve in approximately the given position. The exact shape of the curve is not important. The parameters should determine the kind of curve and the location. Use your calculator to check your guesses.

Sketch

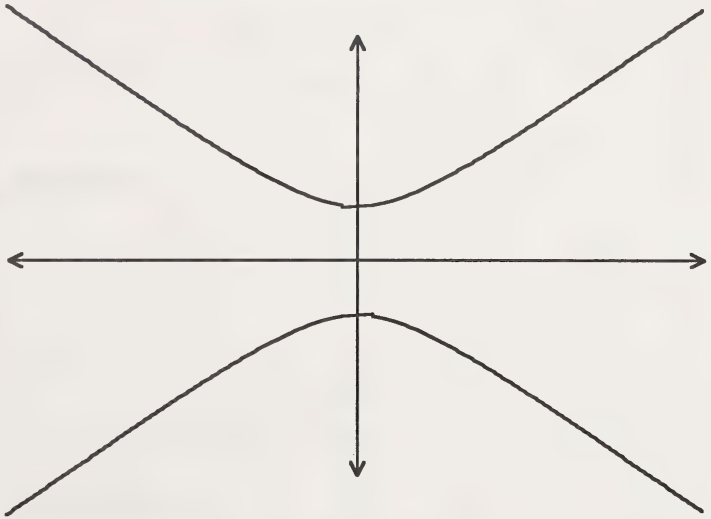


a. $A =$ ____ $B =$ ____ $C =$ ____ $D =$ ____ $E =$ ____

$F =$ ____

Possible equation: _____

Sketch

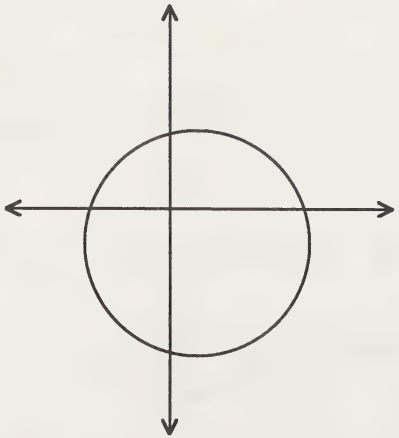


b. $A =$ ____ $B =$ ____ $C =$ ____ $D =$ ____ $E =$ ____

$F =$ ____

Possible equation: _____

Sketch

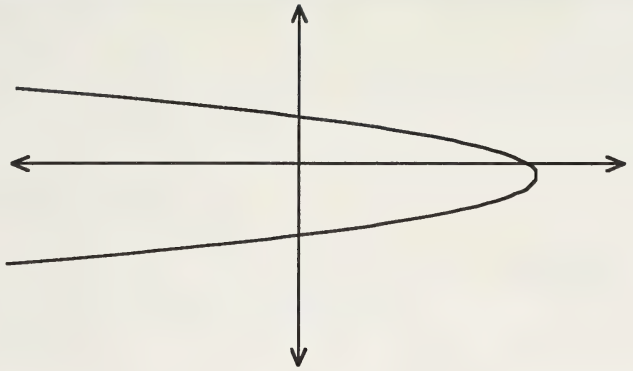


c. $A =$ ____ $B =$ ____ $C =$ ____ $D =$ ____ $E =$ ____

$F =$ ____

Possible equation: _____

Sketch



d. $A =$ ____ $B =$ ____ $C =$ ____ $D =$ ____ $E =$ ____

$F =$ ____

Possible equation: _____

For solutions to **Activity 5**, turn to **Appendix A, Topic 2.**

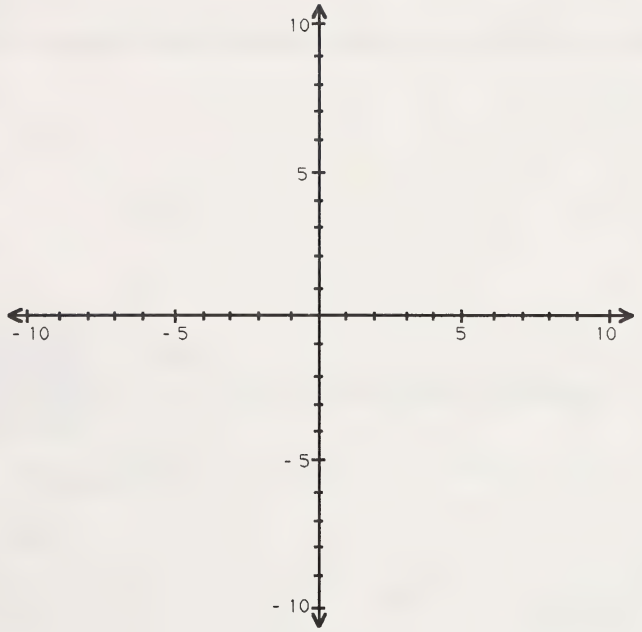
Activity 6

Understand some of the effects of changing B in $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

In Investigation 2 you saw some of the effects of changing the value of B on the type of graph that is produced. You saw that changing the value of B can give you a circle, a parabola, an ellipse, or a hyperbola. In this activity you will look at some of the special effects that changing the value of B can have on the curves.

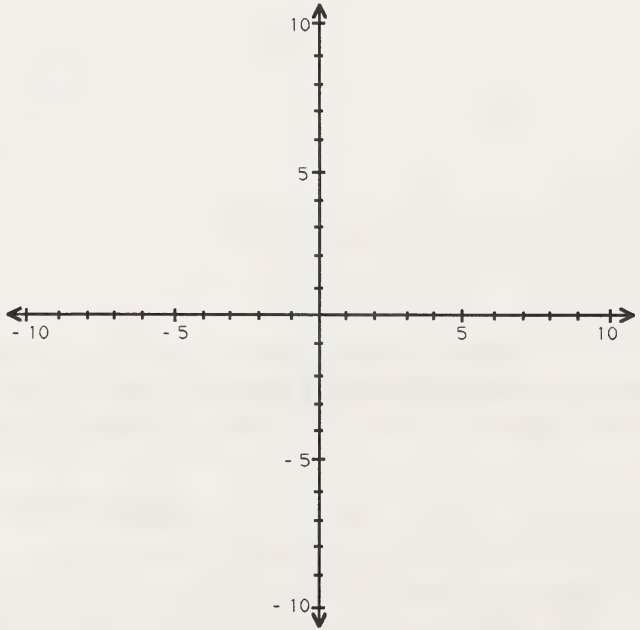
Investigation 19

Using the equation $x^2 + Bxy + y^2 - 25 = 0$, try $B = 1$. (Note that $A, B, C = 1$.) Draw the resulting graph on the following grid. Now make $B = -1$ and draw the resulting graph on the grid.



Observations:

Using the equation $Ax^2 + Bxy + Cy^2 - 25 = 0$, try $A, B, C = 2$. Now change B to -2. Draw the graphs. Also try $A, B, C = 3$. Now change B to -3. Draw the graphs, and answer the question which follows.



Observations:

What can you conclude about the effect of B on the ellipse?

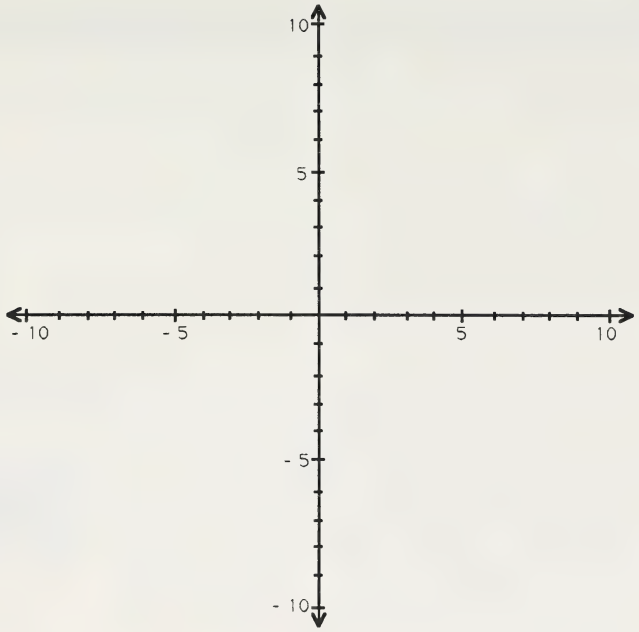
Conclusion:

_____

_____

Investigation 20

Using the equation $Bxy - 25 = 0$, try $B = 1$ and $B = -1$. Graph the results and answer the questions which follow.



Observations:

a. What are the resulting curves?

_____

_____

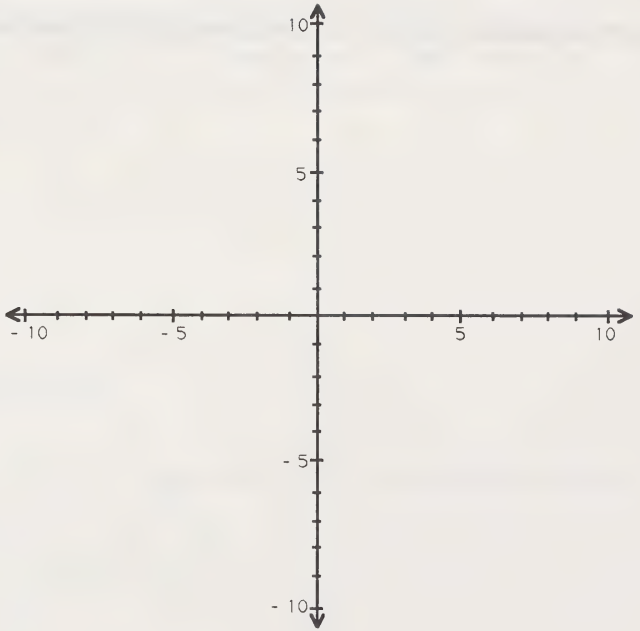
b. What is the effect of changing B from positive to negative?

_____

_____

Investigation 21

In this investigation you will learn how the shape of the graph changes as B changes. Use the equation $x^2 + xy + 9y^2 - 9 = 0$. In this equation we have $A = 1$, $B = 1$, $C = 9$, $D = 0$, $E = 0$, and $F = -9$. Use the values specified for B in each question.



Observations:

a. Now graph the equation with $B = 1$. Draw the graph in the preceding grid. What type of graph do you get?

b. Now graph the equation with $B = 4$. Draw the graph on the grid. Now what has happened to the graph?

c. Graph the equation with $B = 7$. What type of graph do you get?

d. Graph the equation with $B = 5$. What type of graph do you get?

e. Graph the equation with $B = 6$.

i. What type of graph do you get?

ii. What is the relationship between A, B, and C when $B = 6$?

Did you find that the values of A, B, and C make the equation a perfect square? Notice that $x^2 + 6xy + 9y^2$ can be factored to $(x + 3y)(x + 3y)$. In this case the graph is two straight lines. This is another degenerate conic. Remember a degenerate conic can either be two parallel lines, two intersecting lines, a single straight line, or a point.

f. Repeat the investigation using $B = -1, -4, -5, -6$, and -7. How do these graphs compare with the graphs where B was positive?

g. Try other combinations of A, B, and C which make a perfect square. Graph these on your calculator. Do you get the same results?

_____

h. Try $x^2+6xy+9y^2=0$. Notice this time that $F=0$. What is the result? Can you explain the result algebraically?

_____

_____

_____

_____

_____

What can you conclude about the effect of changing the value of B on the graph of the conic? Summarize the effects of changing B here.

Summary:

_____

_____

_____

_____

_____

For solutions to **Activity 6**, turn to **Appendix A, Topic 2.**

This is the end of the investigations on the changes that occur when the parameters are changed. As you are probably aware by now, the number of changes that can be investigated is almost without limit. If you wish, you may continue with various investigations on your own.

If you have access to a videocassette player, you may wish to view the program **Graphing the Curves**. This is program 2 in the series *Discovering Conics*. This video shows the various curves produced and the changes on the curves when the parameters (A, B, C, D, E and F) are varied in the general equation of the quadratic.

Topic 3 Defining Each Conic as a Locus of Points

Introduction

In this topic you will be given activities that will require you to draw graphs of conics. From these graphs you will write a rule or definition of the locus of various conic sections. Subsequent activities will lead you from the locus definition to the equation of the curve. Once this equation has been obtained, you should be able to verify the locus definition by picking points on the locus. Your knowledge of the locus will help you solve some problems that involve applications.

What Lies Ahead

Throughout this topic you will learn to

1. recognize that each conic can be described as a locus of points

Now that you know what to expect, turn the page to begin your study of defining each conic as a locus of points.

Exploring Topic 3

Activity 1

Recognize that each conic can be described as a locus of points.

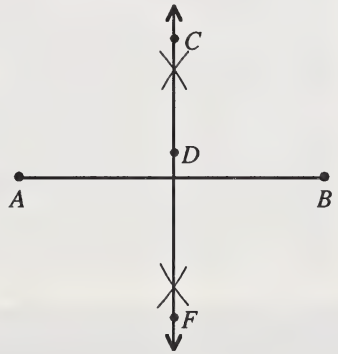
The paths of points can be described by some rule that defines all points on the curve. The following example will show you how to establish this rule.

Example 1

Choose two points, A and B, on a sheet of paper. Find all of the points that are equidistant from these two points.

Solution:

Draw a line segment AB.
Using a compass, construct the perpendicular bisector of $\overline{AB}$. Locate points C, D, and F at random on this line.



$d(CA) = d(CB)$
$d(DA) = d(DB)$
$d(FA) = d(FB)$

You will notice that the points on $\overline{CD}$ are equidistant from A and B. You can write a rule for all the points on this line. The rule is as follows: A set of points are equidistant from two fixed points.
A locus is a path or a set of points that follow a rule. This rule is called the **locus definition**. A locus definition for this example is as follows: The perpendicular bisector of $\overline{AB}$ is the locus of all points equidistant from two fixed points A and B.

Each conic has its own locus definition. You will discover these definitions through investigations on the different conics. Do the investigations first; then, turn to **Appendix A** for the correct solutions.

Investigation 1

Draw a circle with radius 4 cm and centre C. Next, write a rule that will define all points on the curve and then give the locus definition for the set of points on the curve.

Investigation 2

Using a compass and a sheet of ordinary lined paper, construct a figure by completing the following steps.

Step 1: At the bottom of the paper draw a line D.

Step 2: Label the lines above D as 1, 2, 3, and so on. The space between two adjacent lines on the paper measures 1 unit.

Step 3: On line 2 mark any point F.

Step 4: On line 1 label point V, 1 unit from D and 1 unit from F. (V is the point of intersection of line 1 and the perpendicular line segment from F to D.)

Step 5: With the compass point on F and a radius of 2 units from D, mark two points on line 2. With the compass point on F and a radius of 3 units from D, mark two points on line 3. (Remember that three spaces between the lines will give you 3 units of measure.)

Continue this process until you get a reasonable number of points so that you can see the shape of the curve.

Step 6: Join the points with a smooth curve. What shape results?

Step 7: Choose a point P_1 on the curve. Measure $d(P_1F)$ and $d(P_1D)$. How do the distances compare? Choose other points on the curve, and find their distances to F and D. What is your conclusion? Write a rule that will define all points on the curve.

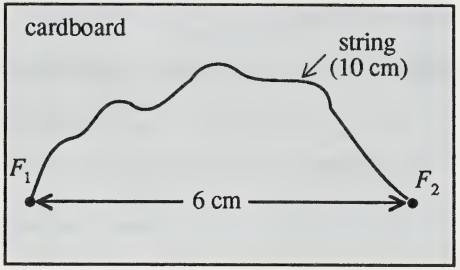
Step 8: The fixed point F is the **focus**, and the fixed line D is called the **directrix** of this curve. Write a locus definition for the set of points on this curve.

Investigation 3

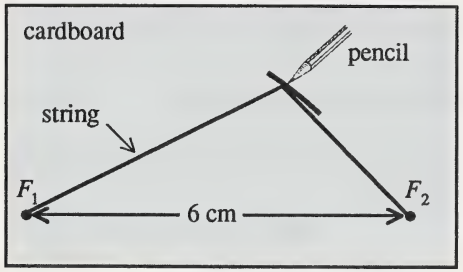
You will need a piece of cardboard about 40 cm × 30 cm and a piece of string about 20 cm long. Complete the following.

Step 1: Make two holes about 6 cm apart in the cardboard. Label the holes F_1 and F_2.

Step 2: Pass an end of the string through each of the two holes and tie a large knot at each end so that the length of the string on the cardboard measures 10 cm.



Step 3: Keeping the string taut, move the pencil to trace a curve.



a. What shape is the curve?

b. Choose points A, B, and C anywhere on the curve. Find the following measures with a ruler.

i. $d(AF_1)+d(AF_2)$

ii. $d(BF_1)+d(BF_2)$

iii. $d(CF_1)+d(CF_2)$

c. What is your conclusion?

d. Write a rule that will define all points on the curve. (The two fixed points F_1 and F_2 are called the **foci** of this curve.)

e. Give the locus definition for the set of points on this curve.

f. Change the length of the string. What effect does it have on the shape of the curve?

g. Move the holes closer to each other. What effect does it have on the shape of the curve?

h. If $F_1 = F_2$, what is the shape of the curve?

Investigation 4

On a sheet of paper label two points F_1 and F_2 and space them 8 cm apart. Use a compass to locate two points that are 4 cm from F_1 and 8 cm from F_2. (You will get a point on either side of line segment F_1F_2.)

Then locate two points that are 3 cm from F_1 and 7 cm from F_2, and two points that are 2 cm from F_1 and 6 cm from F_2. Then join all the points. Reverse the process in order to locate two points that are 8 cm from F_1 and 4 cm from F_2. Next locate points that are 7 cm from F_1 and 3 cm from F_2, and two points that are 6 cm from F_1 and 2 cm from F_2. Then join all the points.

a. What conic is created?

b. Choose points P_1, P_2, and P_3 on the curves. Find the following measures.

i. $\left|d(P_1F_1)-d(P_1F_2)\right|$

ii. $\left|d(P_2F_1)-d(P_2F_2)\right|$

iii. $\left|d(P_3F_1)-d(P_3F_2)\right|$

c. What is your conclusion?

d. Write a rule that will define all the points on the curve. (Note: The fixed points F_1 and F_2 are the **focal points** of the curve.)

e. Give the locus definition for the set of points on this curve.

f. What effect will it have on the curve if the distance between the foci is decreased and the constant of the difference is fixed? What restriction applies?

In the next two investigations you will draw the curve of quadratic relations from the locus definition.

Investigation 5

Use the double concentric circle graph paper in **Appendix B** to draw the following curve. Complete these steps.

Step 1: Label the two centres A and B.

Step 2: Select a number. Keep the number less than 20 but more than $d(AB)$. Choose, for example, 16.

Step 3: Find a point P such that the sum of $d(PA)$ and $d(PB)$ is equal to the selected number. For example, if you use the point $(10, 6)$, you see that $10+6=16$.
To locate a point, find the intersection of the concentric circles from A and B. For example, the point $(10, 6)$ is the intersection of the tenth concentric circle from A and the sixth concentric circle from B.

Step 4: Find other points that satisfy this condition.

Step 5: Continue this process until you can draw the complete curve.

a. What shape is this curve?

b. Why does this procedure produce this curve?

Investigation 6

Use the double concentric circle graph paper in **Appendix B** to draw the following curve.

Step 1: Label the two centres A and B.

Step 2: Select a number beetween 1 and 20.

Step 3: Find a point P such that $|d(PA)-d(PB)|$ is equal to the number you selected in Step 2.

Step 4: Find other points that satisfy this condition.

Step 5: Continue this process until you can draw the complete curve.

a. What shape is this curve?

b. Why does this procedure produce this curve?

You will now be able to find the equation of each conic you studied from its locus definition and its graph. Study the following example.

Example 2

Determine the equation for the locus of all points which are 5 cm from a fixed point $C(3,4)$.

Solution:

Let point $P(x,y)$ be any point on the circle. The distance formula is used to find the expression for the distance from point P to the centre of the circle.

This distance is the radius of the circle. The distance formula is as follows:

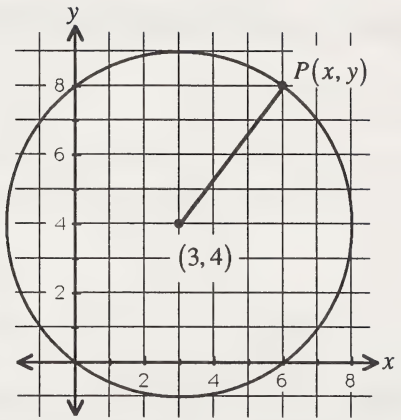
$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

In this problem d is the radius (5 cm).

$$\sqrt{(x-3)^2+(y-4)^2}=5$$

If both sides are squared, the result is as follows:

$$(x-3)^2+(y-4)^2=25$$
$$x^2-6x+9+y^2-8y+16=25$$
$$x^2+y^2-6x-8y=0$$



In Topic 2 you discovered that for a circle centred at the origin $A=C$ and $B=D=E=0$.

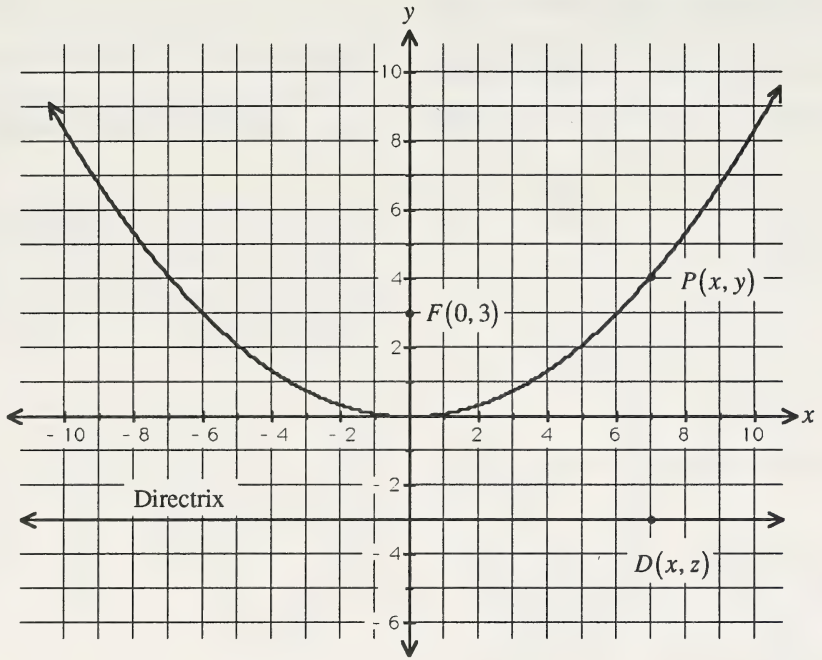
The circle here with centre $(3,4)$ passes through the origin. Therefore, $F=0$.

The locus with centre $(3,4)$ and radius 5 cm is a circle defined by $x^2+y^2-6x-8y=0$.

Now do the following investigations.

Investigation 7

$P(x, y)$ is any point on the following parabola. $D(x, z)$ is a point on the directrix. The coordinates of F are $(0, 3)$.



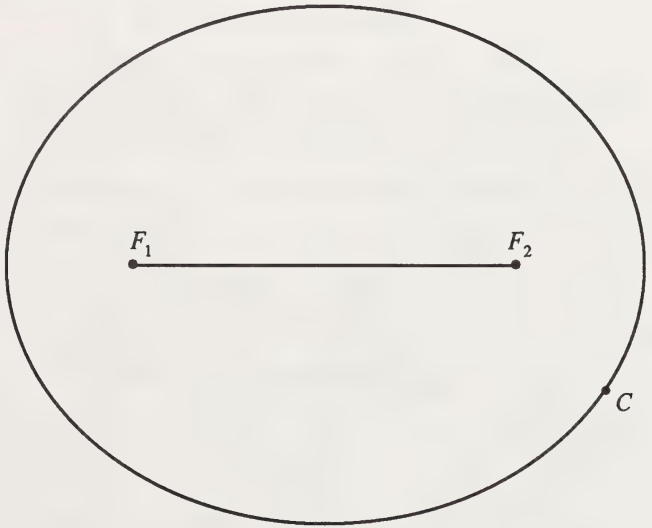
a. What is the value of z?

b. Write the equation of the directrix.

c. Use the locus definition to determine the equation of this parabola.

Investigation 8

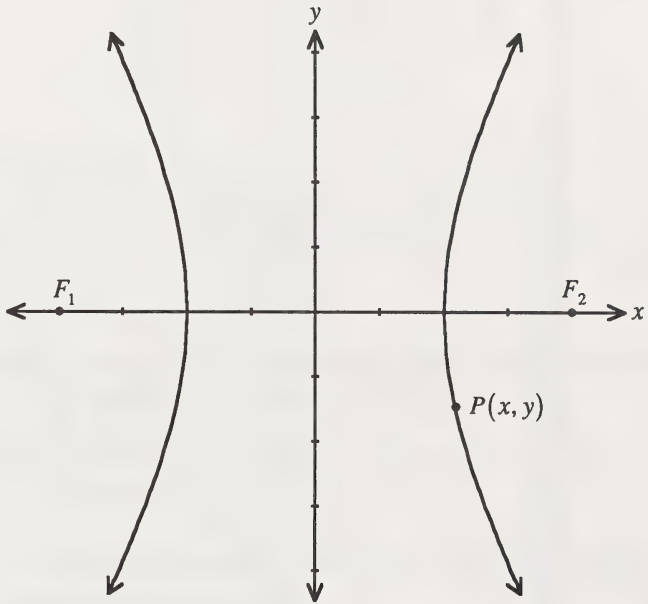
The following is the sketch of the ellipse that was constructed in Investigation 3.



Draw a set of axes so that the ellipse is centred at the origin. Label the coordinates of the two focal points F_1 and F_2. Note that for this curve $d(CF_1)+d(CF_2)=10$. Determine the equation of this ellipse.

Investigation 9

The following diagram represents a sketch of the curve that was drawn in Investigation 4.



Label the coordinates of the foci, and write the equation of this hyperbola if $|d(PF_1)-d(PF_2)|=4$.

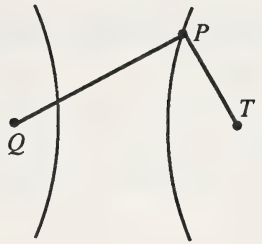
You can verify the locus definition of a curve if its equation and other related information is known. You will discover this in the following investigation.

Investigation 10

The equation of a curve with directrix $y=-4$ and focus $(0,2)$ is $x^2-12y-12=0$. By selecting two points on the curve, verify its locus definition.

You now have the knowledge to solve some problems on applications.

A useful application of the hyperbola is in the navigation of ships and planes. The loran system of navigation is used for determining the locations of ships and planes. Radio signals are sent by two sending stations Q and T which are located as shown in the following diagram. Point P represents the location of a plane on one branch of the hyperbola.



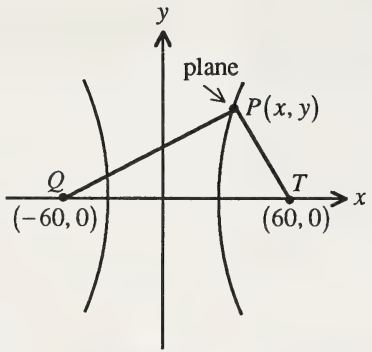
The time difference in the signals is changed to a distance difference $|d(PQ)-d(PT)|$ by computers on the plane. This distance difference is always a constant value. The focal points are Q and T.

The following example illustrates the loran system.

Example 3

Two loran sending stations are 120 km apart. A plane is 40 km closer to station T than it is to station Q. Determine the equation of the hyperbolic path on which the plane is located.

Solution:



The sending stations can be located on either the x- or y-axis. In this example the x-axis will be used as shown in the diagram. The coordinates of the sending stations are $(\pm 60, 0)$ since the stations are 120 km apart. The constant difference is 40 km. In other words $|d(PQ)-d(PT)|=40$. According to the distance formula and the locus definition for a hyperbola, you can determine the equation of the hyperbolic path.

$$|d(PQ)-d(PT)|=40$$

$$\sqrt{[x-(-60)]^2+(y-0)^2}-\sqrt{(x-60)^2+(y-0)^2}=40$$

$$\sqrt{(x+60)^2+y^2}=40+\sqrt{(x-60)^2+y^2} \quad \text{(Square both sides.)}$$

$$x^2+120x+3600+y^2=1600+80\sqrt{(x-60)^2+y^2}+x^2-120x+3600+y^2$$

$$240x-1600=80\sqrt{(x-60)^2+y^2} \quad \text{(Divide each term by 80.)}$$

$$3x-20=\sqrt{(x-60)^2+y^2} \quad \text{(Square both sides.)}$$

$$9x^2-120x+400=x^2-120x+3600+y^2$$

$$8x^2-y^2-3200=0$$

The equation of the hyperbola is $8x^2-y^2-3200=0$.

Now do the following questions.

1. Describe the locus for each of the following.

   a. a seat on a revolving ferris wheel
   b. a spider at the end of a swinging pendulum
   c. the hub of a wagon wheel on a flat road

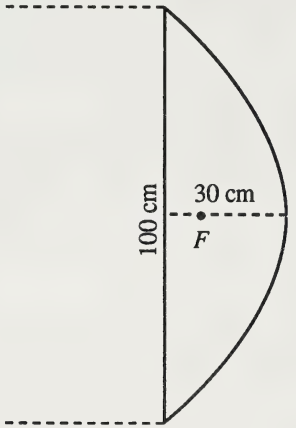
2. A locus of points is given by all the points two units from a fixed point $(-2, 3)$. Is $(-4, 3)$ on the locus?

3. $F_1(-4, 0)$ and $F_2(4, 0)$ are two fixed points. If all the points on the locus satisfy $|d(PF_1) - d(PF_2)| = 6$, show that $(9, 6)$ does not lie on the curve.

4. Define the equation of the ellipse when the foci are $(-2, 3)$ and $(2, 3)$, and the constant sum is 8.

5. A curve with foci $(-4, 0)$ and $(4, 0)$ is defined by the equation $9x^2 + 25y^2 - 225 = 0$. By selecting two points on the locus, verify the locus definition for this curve.

6. A searchlight reflector is designed so that a cross section through its axis is a parabola and the light source is at the focus. The diameter of the reflector is 100 cm and the reflector is 30 cm deep. Find the distance from the vertex to the light source.



For solutions to **Activity 1**, turn to **Appendix A, Topic 3.**

If you have access to a videocassette player, you may find it helpful to view the program **Defining the Locus**. This is program 3 in the series *Discovering Conics*. This video gives the locus definition and shows how to draw the graphs of a parabola, an ellipse, and a hyperbola using concentric-circle graph paper and the locus definition for each curve.

Topic 4 Using Eccentricity to Define Conics

Introduction

You have learned three different ways to define conics. In this topic you are going to learn another way to define conics. Conics will be defined according to different eccentricities. What is eccentricity? You will have to study this topic to find the answer.

What Lies Ahead

Throughout this topic you will learn to

1. describe conics as the locus of points such that the ratio of the distance between any point on the conic and a fixed point (focus) to the distance between the same point and a fixed line (directrix) is a constant

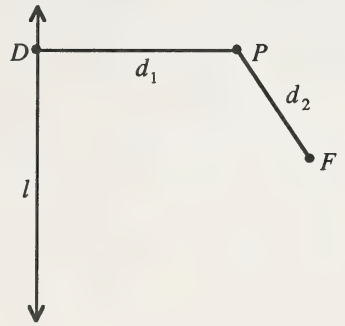
Now that you know what to expect, turn the page to begin your study of using eccentricity to define conics.

Exploring Topic 4

Activity 1

Describe conics as the locus of points such that the ratio of the distance between any point on the conic and a fixed point (focus) to the distance between the same point and a fixed line (directrix) is a constant.

If l is a fixed line (directrix), F is a fixed point (focus) not on that line, and P is any point, it is possible to measure the distance from P to l (distance PD) and from P to F (distance PF). If the ratio of the distance between P and F to the distance between P and l is a constant, then the locus of this point P is a conic.



There is a special name for the point, the line, and the ratio. The fixed line is called a directrix, the fixed point is called a focus, and the ratio of the two distances ($\frac{d_2}{d_1}$ as shown in the diagram) is called the eccentricity (symbol e).

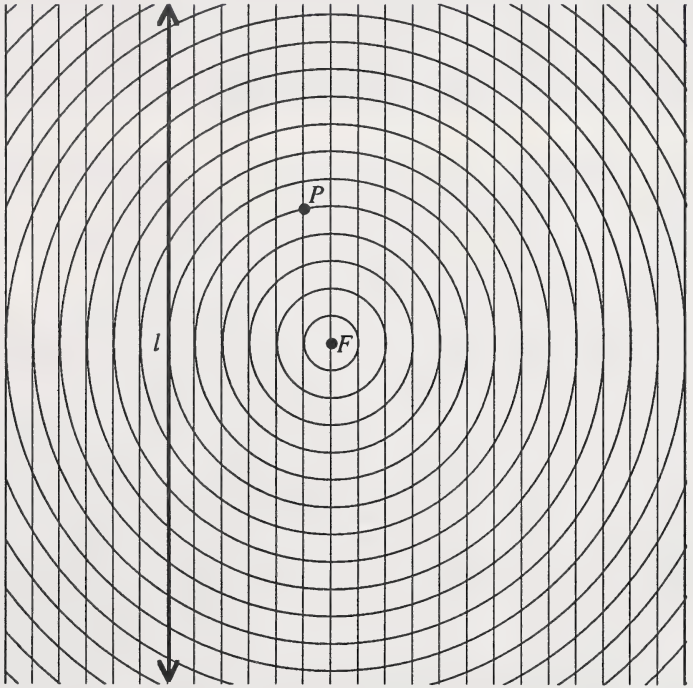
$$e=\frac{d_2}{d_1} \text{ or } e=\frac{d(PF)}{d(PD)}$$

How does eccentricity determine the shape of conics? You can find the answer by doing the following investigations. Answers to the investigations are found in **Appendix A**.

Investigation 1

Use a sheet of circle-line graph paper from **Appendix B** to determine the locus of a point. Let the centre of the concentric circles represent a fixed point (focus), and let the vertical line which is 6 units from the centre represent a fixed line (directrix). Find all the points of the plane with an eccentricity of 1. Do these points determine a conic? What conic is it? (Hint: The radii of the concentric circles are increased by the same unit as the vertical lines. Therefore, if the number of circles from the centre is equal to the number of lines from the directrix, then the point of intersection of the last circle and the last vertical line is the point equidistant from the focus and directrix.)

The following example shows that P is 5 units from F and l.



Investigation 2

Take another sheet of circle-line graph paper. Let the centre of the concentric circles represent the focus, and let any one vertical line represent the directrix. Now repeat Investigation 1. Do you get the same kind of conic?

Investigation 3

Take a sheet of circle-line graph paper from **Appendix B**. Let the centre of the concentric circles represent a fixed point (focus), and choose any one vertical line (try putting this line 6 units from the focus) to represent the directrix. Find any point P such that the ratio of d_2 (the distance from P to F) to d_1 (the distance from P to the directrix) is $\frac{2}{3}$. (Hint: $\frac{2}{3}$ is the same as $\frac{4}{6}, \frac{6}{9}, \frac{8}{12} \cdots \frac{d_2}{d_1}$. Therefore, $d_2 = 4$ when $d_1 = 6$, $d_2 = 6$ when $d_1 = 9$, and so on.) What conic do these points (P) determine?

Investigation 4

Repeat Investigation 3 using different ratios (eccentricity) between 0 and 1. (Hint: You may find it difficult to sketch the graph if $e < \frac{1}{2}$.) Compare the flatness of the ellipses. What other conjectures can be made?

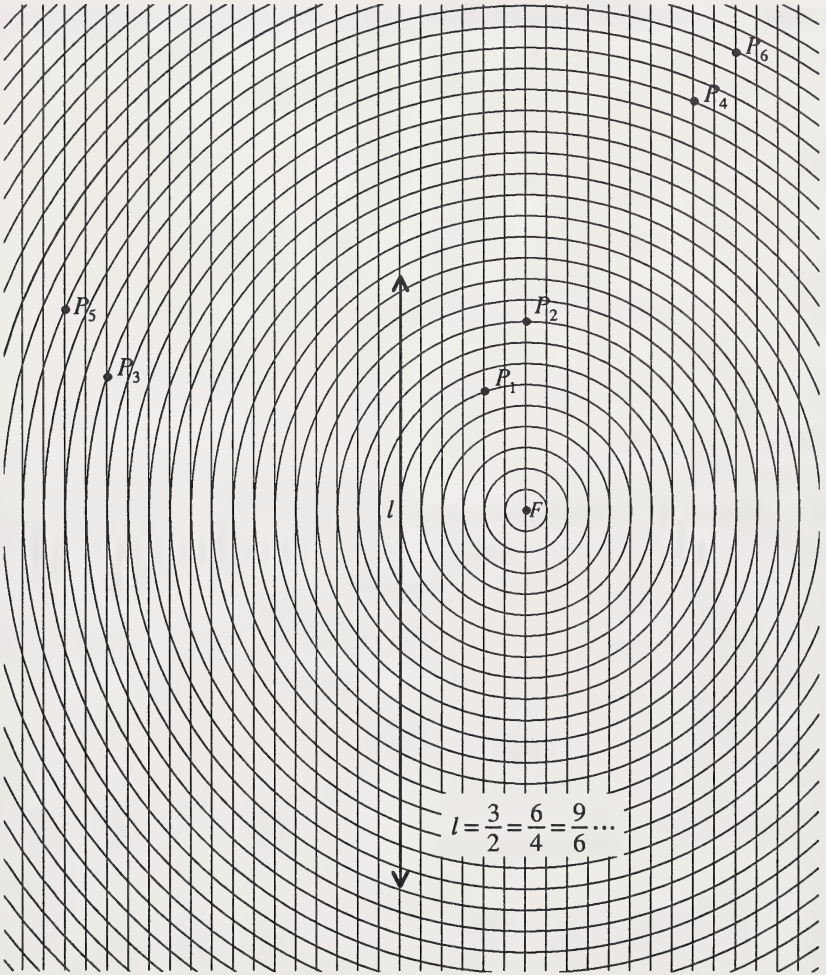
If you have finished Investigation 4, you should be able to tell that the eccentricity measures the flatness of an ellipse. As e approaches 1, the ellipse approaches a line segment. On the other hand, as e approaches 0, the ellipse approaches a circle. In other words, the limiting position of an ellipse is a circle as e approaches 0, and the limiting position of an ellipse is a line segment as e approaches 1. The length of this segment would be twice the length of the radius of the circle. Since the limiting position of an ellipse is a circle as e approaches 0, you can say that a circle is a special ellipse.

Investigation 5

Take a sheet of circle-line graph paper from **Appendix B**. Let the centre of the concentric circles represent the focus. Let $e = \frac{3}{4}$. Use three different vertical lines to represent three different directrices. Sketch the three different ellipses on the same sheet of graph paper. What happens to the ellipses? Summarize your findings in a paragraph.

Investigation 6

Take a sheet of circle-line graph paper from **Appendix B**. Let the centre of the concentric circles represent the focus (F). Choose a vertical line which is 6 units from the focus to represent the directrix l. Find any point P such that the ratio of $d(PF)$ to $d(Pl)$ is $\frac{3}{2}$. What conic do these points (P) determine? (Hints: $\frac{3}{2}$ is the same as $\frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \frac{15}{10}, \frac{18}{12}, \frac{21}{14}, \cdots \frac{d(PF)}{d(Pl)}$. It is also possible to find the points of intersection on both sides of the directrix. The following diagram shows that P_1 is 6 units from F and 4 units from l; P_2 is 9 units from F and 6 units from l; P_3 and P_4 are both 21 units from F and 14 units from l; and P_5 and P_6 are both 24 units from F and 16 units from l. This tells you that there are points on both sides of the directrix. Make sure that you find enough points on both sides of the directrix.)


P_6
P_4
P_5
P_2
P_3
P_1
l
F
$l = \frac{3}{2} = \frac{6}{4} = \frac{9}{6} \cdots$

Investigation 7

Repeat Investigation 6 using the the same focus and same directrix, but different eccentricities which are greater than 1. Sketch all conics on the same sheet of graph paper. What conjectures can be made?

Investigation 8

Repeat Investigation 6 using the same focus and the same eccentricity, but different directrices. Sketch all conics on the same sheet of graph paper. What conjectures can be made?

If you put the focus and directrix back to an ordinary coordinate system, can the equation of the locus be found when eccentricity is given? The answer is in the next investigation.

Investigation 9

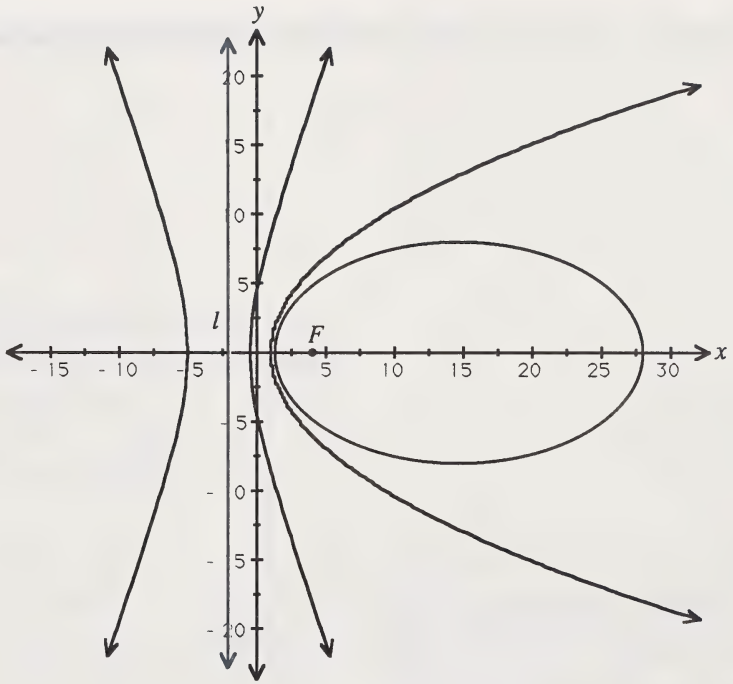
Take a sheet of ordinary graph paper from **Appendix B**. Draw the point $F(2, 3)$ and the vertical line $x = -1$. If $P(x, y)$ is any point on a conic and the ratio of d_2 (the distance from P to F) to d_1 (the distance form P to the vertical line) is $\frac{3}{2}$, use the distance formula and the definition of eccentricity to find the equation of the locus. What conic does this equation represent?

Investigation 10

Repeat Investigation 9. Use the same directrix and focus. First change eccentricity to 1; then change it to $\frac{3}{4}$. After finding the equations of the conics, use a computer or a graphic calculator to sketch the graphics. Use eccentricity to define the three conics, namely the parabola, ellipse, and hyperbola.

Investigation 11

The following graph shows a parabola, an ellipse, and a hyperbola. They have the same focus and directrix. Choose a point on each conic and use this point to determine the eccentricity of the conic. Check your answer by choosing another point and repeating the calculation.



Now do the following questions.

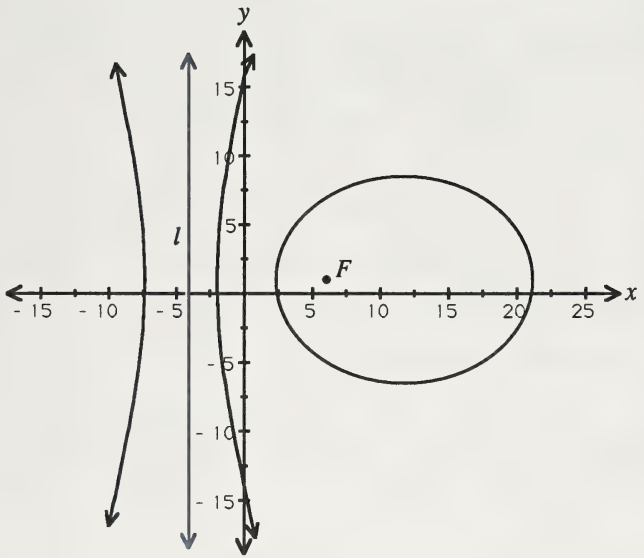
1. The eccentricity of a parabola is always equal to 1. Does it mean that the eccentricity does not affect the shape of a parabola? Identify and explain how the other factors affect the shape of a parabola.

2. How does the eccentricity affect the shape of an ellipse?

3. What is the effect of a smaller eccentricity on the shape of a hyperbola?

4. The vertical line $x = 3$ is a directrix. The point $F(5, 2)$ is the focus. The eccentricity is $\frac{4}{5}$. Find the equation of the conic. What conic is it?

5. The following graph shows an ellipse and a hyperbola. Determine the eccentricities of the conics. (l is the directrix and F is focus for each conic.)



6. a. The distance between a directrix and a focus is 4 units. Use a circle-line graph to sketch the conic which has an eccentricity of 3.

   b. What conic is it?

7. The vertical line $x = \frac{16}{5}$ is a directrix. The point $P(5, 0)$ is a focus. The eccentricity of the conic is $\frac{5}{4}$. Prove that the point $(4, 0)$ is a point on the conic?

For solutions to **Activity 1**, turn to **Appendix A, Topic 4.**

If you have access to a videocassette player, you may wish to view the program **Eccentricity**. This is program 4 in the series *Discovering Conics*. This video shows how to draw a parabola, ellipse, or hyperbola using concentric-circle graph paper and using a given eccentricity value.

Unit Summary

What You Have Learned

In this unit you have studied quadratic relations in four different ways.

- You have studied the conics that are generated when a plane cuts a double-napped cone in various ways.
- You have studied what happens when you change the parameters in the general quadratic relation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. These changes are discussed through changing various combinations of $A, B, C, D, E,$ or F.
- You have studied using the locus definition to define the conics and to verify the equations that describe the conics.
- You have studied using eccentricity to define the conics where the eccentricity is the ratio of the distance between any point and the focus to the distance between the same point and the directrix.

The four main conics are the circle, ellipse, parabola, and hyperbola. There are also five degenerate conics that can be shown: two parallel lines, two intersecting lines, one line, one point, or no locus.

You are now ready to

complete the **Unit Assignment**.

Appendices

Appendix A
Solutions

Review

Topic 1 The Physical Properties of Conic Sections

Topic 2 Describing Conics Through the Parameters

Topic 3 Defining each Conic as a Locus of Points

Topic 4 Using Eccentricity to Define Conics

Appendix B
Graphing Materials

Grids

Circle-Line Graphs

Double Concentric Circle Graphs

Appendix A
Solutions

Review

1. $d(AB) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(-5-6)^2 + [7-(-5)]^2}$

$= \sqrt{(-11)^2 + (12)^2}$

$= \sqrt{121+144}$

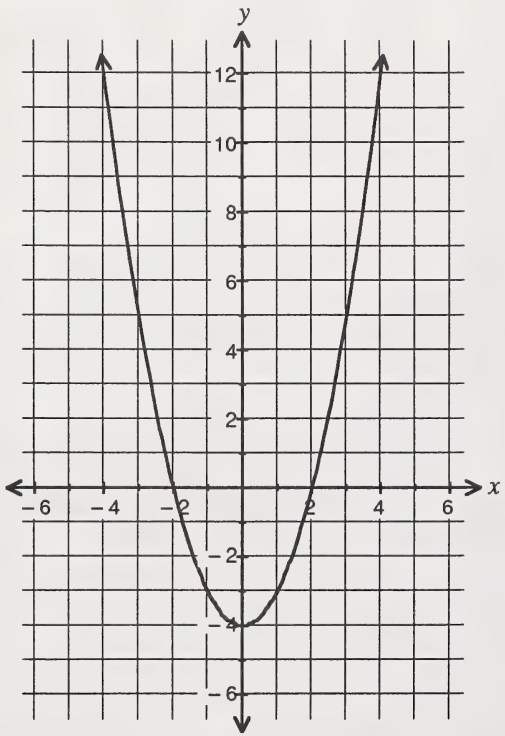
$= \sqrt{265}$

$\doteq 16.28$

2. a. The vertex is at $(0, -4)$.
   b. The axis of symmetry is $x = 0$.
   c. The x-intercepts are -2 and 2.
   d. The y-intercept is -4.
   e. The minimum value is -4.

$y = x^2 - 4$

x	y
-4	12
-3	5
-2	0
-1	-3
0	-4
1	-3
2	0
3	5
4	12



3. The graph on the calculator should look similar to the one that was drawn in question 2.

4. a. quadrant 3
   b. quadrant 1
   b. quadrant 2
   d. quadrant 4

5. a. The graph moves down 5 units.
   b. The graph move 5 units to the left.
   c. The graph moves up 5 units and reflects it in the x-axis.
   d. The graph moves down 5 units and reflects it in the x-axis.

Exploring Topic 1

Activity 1

Demonstrate an understanding of the physical properties of the conic sections with respect to the intersection of a plane and a double-napped cone.

1. You can produce a circle in the following ways:

   - drawing a circle
   - cutting out a circle
   - pouring liquid into an erect cone or cylinder
   - mapping or printing using a circular object
   - cutting a cone
   - others

2. You can produce a parabola in the following ways:

   - cutting a cone at an angle
   - by firing a missile up in the air and allowing it to fall
   - rolling a ball up a smooth slope and mapping its path
   - others

3. You can produce an ellipse in the following ways:

   - cutting a cylinderical pipe at an angle
   - pouring a liquid substance into a tire
   - cutting a cone at an angle
   - looping a string around two fixed pins and mapping a path
   - others

4. You can produce a hyperbola in the following ways:

   - cutting a right double cone with a vertical blade
   - drawing a special curve and reflecting it in a mirror
   - others

5. a. When you make a vertical cut, you get a hyperbola.
   b. When you make a horizontal cut, you get a circle.
   c. When you makle a slant cut, you get an ellipse.

6. When you make a vertical cut not passing through the centre, you get a hyperbola.

7. a. When you slice it perpendicular to the vertical axis, you get two parallel lines.

   b. When you slice it at an angle to the axis, you get an ellipse.

8. a. When you hold it perpendicular to the wall, you get a circle.

   b. When you hold it at an oblique angle, you get an ellipse.

   c. When you move closer to the wall, the light curve decreases in size.

d. When you move away from the wall, the light increases in size.

9. a. When the solution is held horizontally, the shape in the cup is circular.

   b. When the cup is tilted at an angle, the shape in the cup is elliptical.

10. a. When you pour a litre of water into a car tire, the shape you will observe is an ellipse.

    b. When you cut a pipe at an angle of 45°, you will notice that the cross section forms a normal ellipse.

    c. If you cut the pipe at various angles, you will notice that when the angle is less than 45°, the ellipse you observe will have a greater degree of curvature. When the angle is greater than 45°, the ellipse you get will be more elongated.

11. a. When you flatten the dixie cup and cut a slice, you get a V-shape almost similar to the flattended cup.

    b. When you cut off a slice parallel to the generator or side of the cup, you get a parabolic shape.

12. In order to produce the conic sections, you need to have two right circular cones of equal size. You then connect the vertices in such a way that the bases of the cones are pointing in opposite directions.

    Use a plane to make various cuts vertically, horizontally, and obliquely.

13. It is often said that a circle is the limiting case of an ellipse. The reason for this is that when you cut a cone at an angle, the cross section will depict an ellipse. If you make the angle of the cut decrease towards the horizontal, the ellipse will get more and more rounded. Eventually, when the angle is zero you will have reached the limiting position of the ellipse. In other words it changes from an ellipse to a circle. The point at which the ellipse ceases to be an ellipse and becomes a circle is called the limiting position of the ellipse.

14. If a circle were drawn on a thin rubbery material, its shape would be changed when a force is applied pulling the rubbery sheet apart. The sheet would stretch and the circle would become an ellipse. Of course the greater the pull, the more the elongation of the ellipse. We have to assume that the pull is uniform and evenly distributed.

Exploring Topic 2

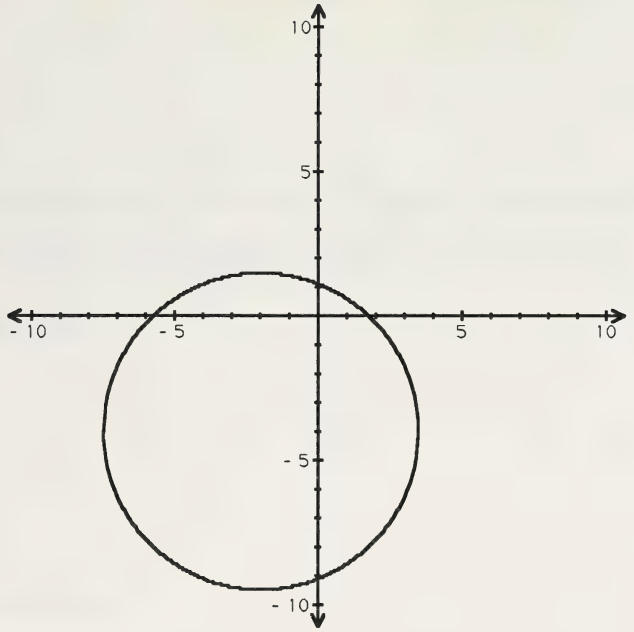
Activity 1

Understand some of the effects of changing the numerical coefficients one at a time in the equation

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$

Investigation 1

a.

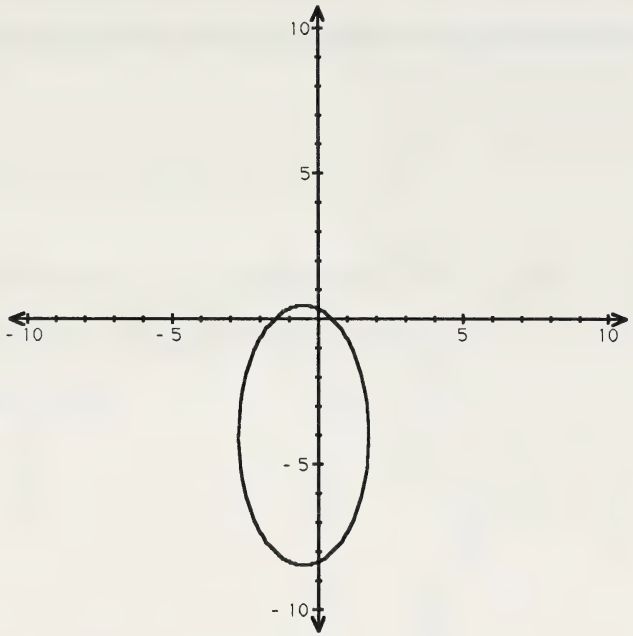


Observations:

The graph is a circle with a centre in quadrant 3.

The coefficients are $A = 1$, $B = 0$, $C = 1$, $D = 4$, $E = 8$, and $F = -10$.

b.

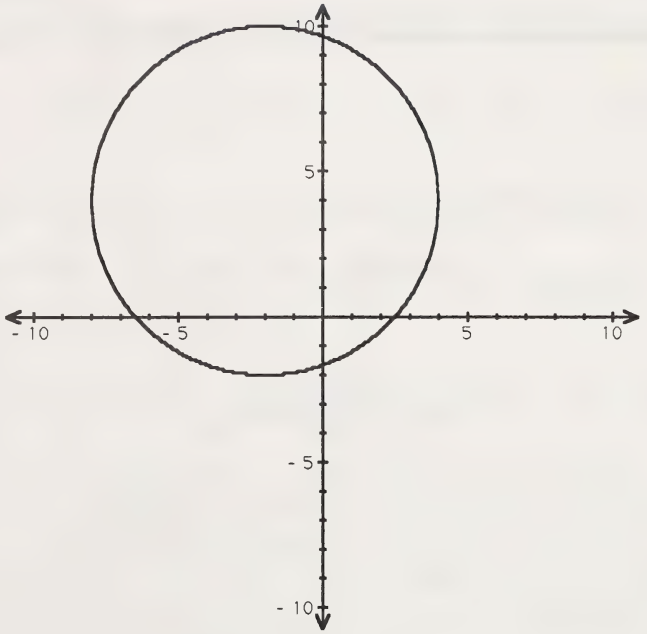


Observations:

The graph is an ellipse with centre in quadrant 3.

The coefficients are $A = 4$, $B = 0$, $C = 1$, $D = 4$, $E = 8$, and $F = -3$.

c.

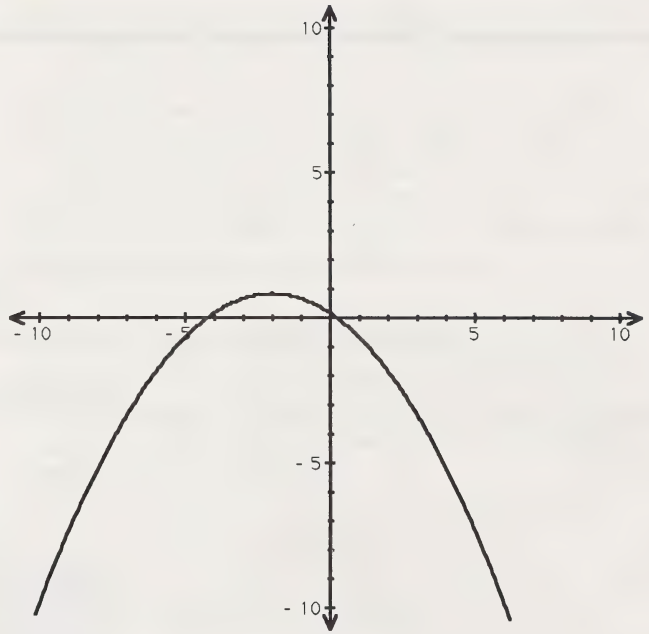


Observations:

The graph is a circle with centre in quadrant 2.

The coefficients are $A=1$, $B=0$, $C=1$, $D=4$, $E=-8$, and $F=-16$.

d.

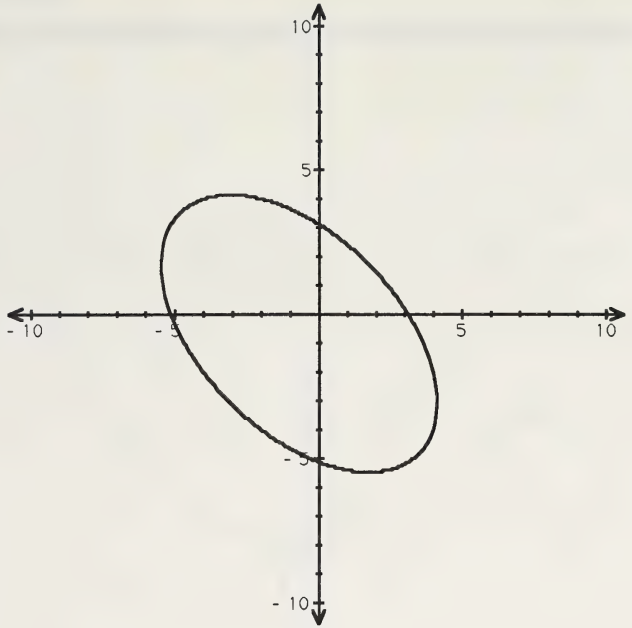


Observations:

The graph is a parabola with vertex in quadrant 2.

The coefficients are $A=1$, $B=0$, $C=0$, $D=4$, $E=6$, and $F=-1$.

e.

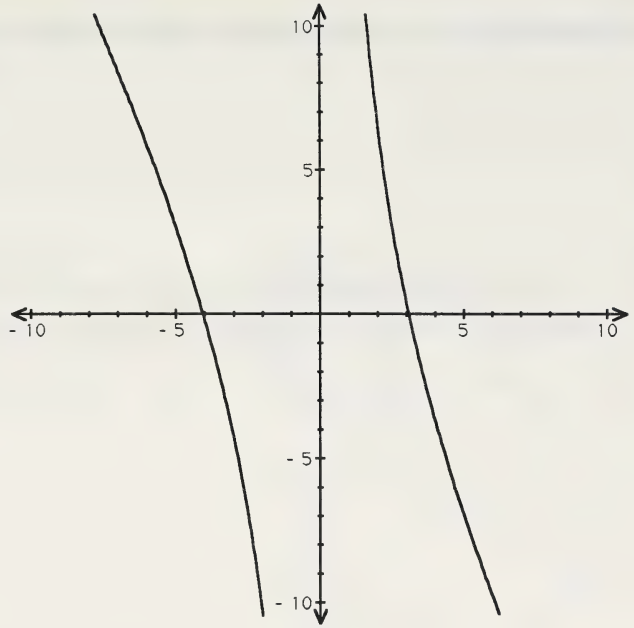


Observations:

The graph is an ellipse with centre in quadrant 3. The graph is rotated 45° from the axis.

The coefficients are $A=1$, $B=1$, $C=1$, $D=2$, $E=2$, and $F=-16$.

f.



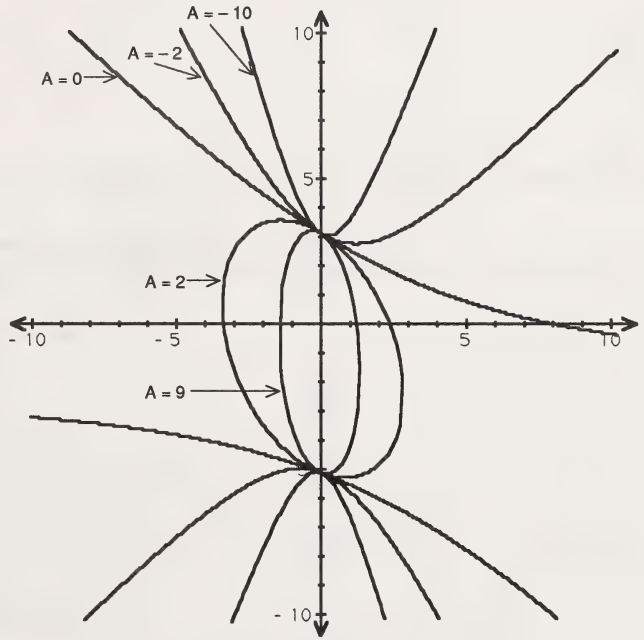
Observations:

The graph is a hyperbola with an axis of symmetry rotated slightly left of vertical.

The coefficients are $A=2$, $B=1$, $C=0$, $D=2$, $E=0$, and $F=-25$.

Investigation 2

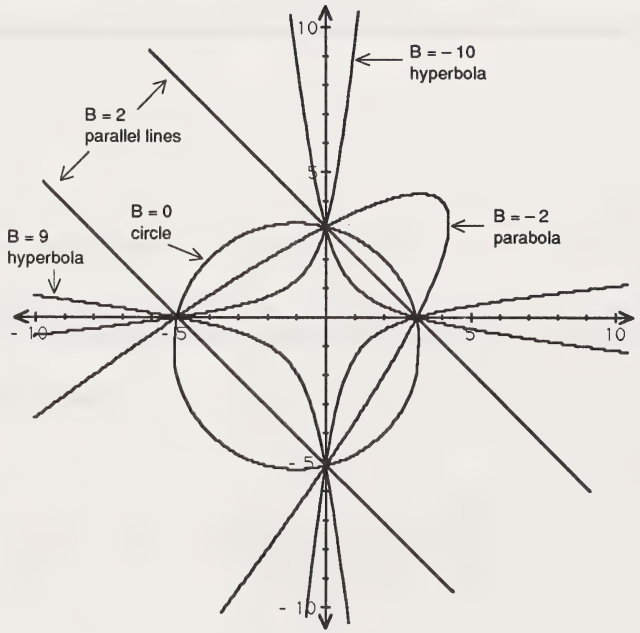
a.



Observations:

As A changes from -10 to 9, the curve changes from hyperbolas (when A is zero or negative) to ellipses (when A is positive).

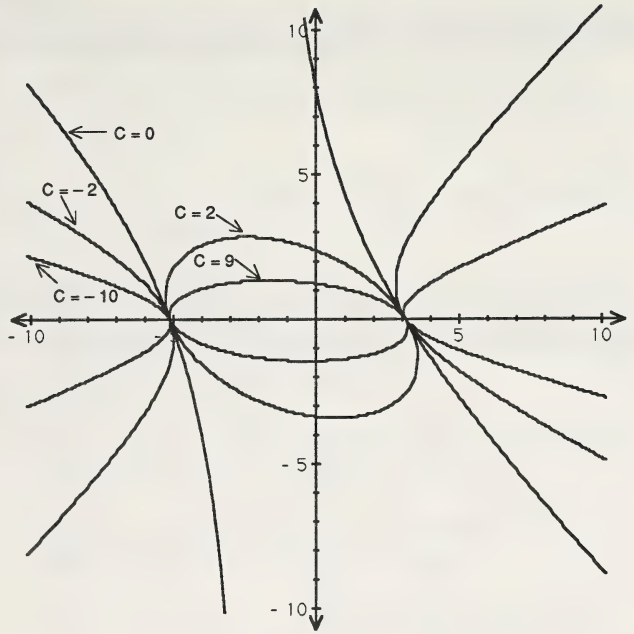
b.



Observations:

As B changes from -10 to 9, the curve changes from a hyperbola, to a parabola, to a circle, to two straight lines, and back to a hyperbola.

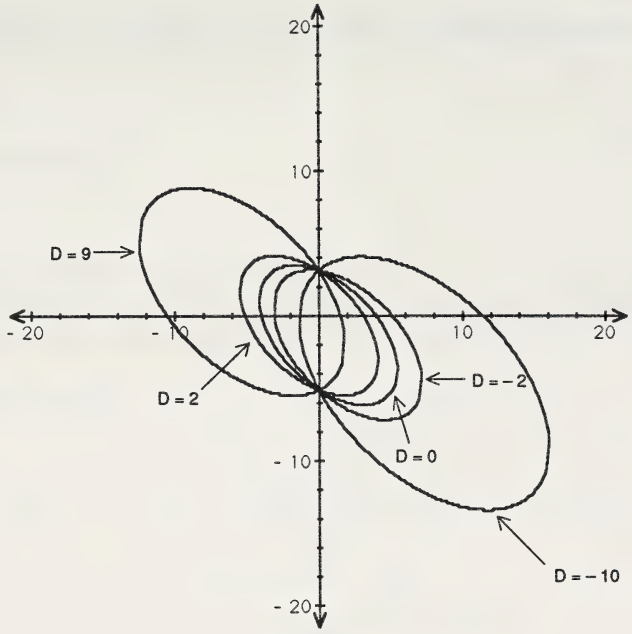
c.



Observations:

As C changes from -10 to 9, the curve changes from hyperbolas to ellipses. The changes are similar to those that occurred when A was changed, except the curves are not along the x-axis.

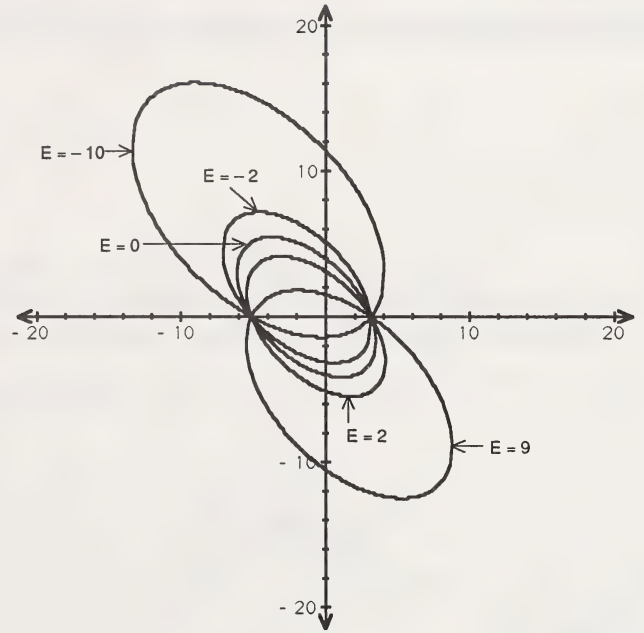
d.



Observations:

As D changes from -10 to 9, the curve moves from having its centre in quadrant 4 to having its centre in quadrant 2. The type of curve does not change; however, the size of the curve is changing.

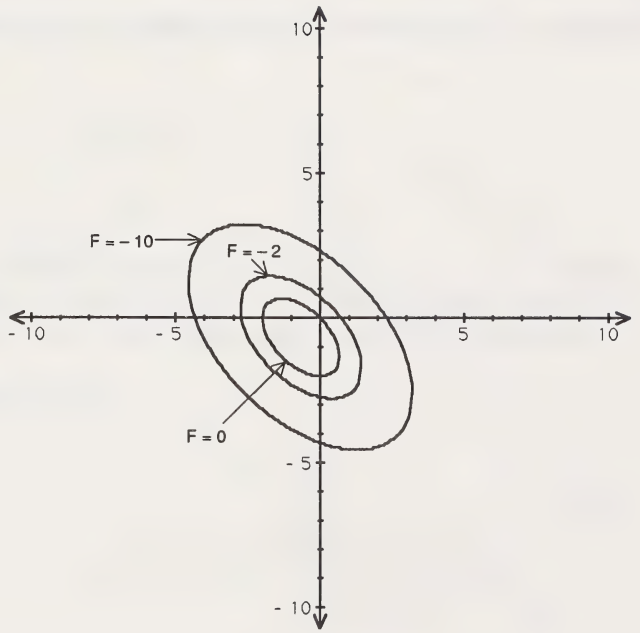
e.



Observations:

As E changes from -10 to 9, the curve moves from having its centre in quadrant 2 to having its centre in quadrant 4. The changes are similar to the changes for D.

f.



Observations:

As F changes from -10 to 0, the curve is getting smaller. When F is greater than zero, no graph is possible. The reasons for this will be discussed in a later investigation.

1. a. Changing the value of A appears to change the type of curves as well as the shape of the curve. Going from positive to negative A changes the type of curve, while changing the magnitude of A with the sign remaining the same changes the shape.

   b. The type of curve appears to change as both the magnitude and sign of B changes.

   c. Changing the value of C changes the type of curve as well as the shape of the curve as C goes from a positive value to a negative value. The effects are similar to those of A, except the changes are along the x-axis.

   d. Changing the value of D appears to affect only the size and position of the curve, not the type or shape.

   e. Changing the value of E appears to affect only the size and positon of the curve, not the type or shape.

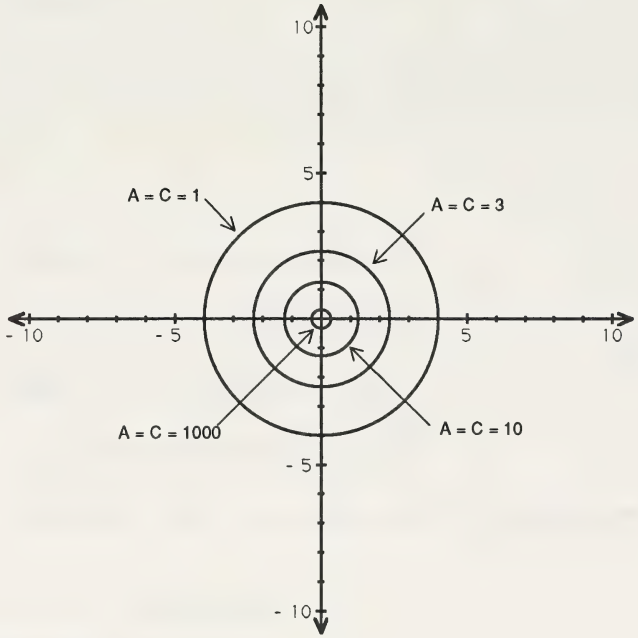
   f. Changing the value of F appears to affect the size of the curve only. However, when F was made positive no curve was generated.

Activity 2

Understand some of the effects of changing A and C in $Ax^2 + Cy^2 + Dx + Ey + F = 0$.

The solutions in this activity contain possible values for the parameters in question. You may have different values, but you should have the same answers to the questions, and you should have similar conclusions.

Investigation 3



Observations:

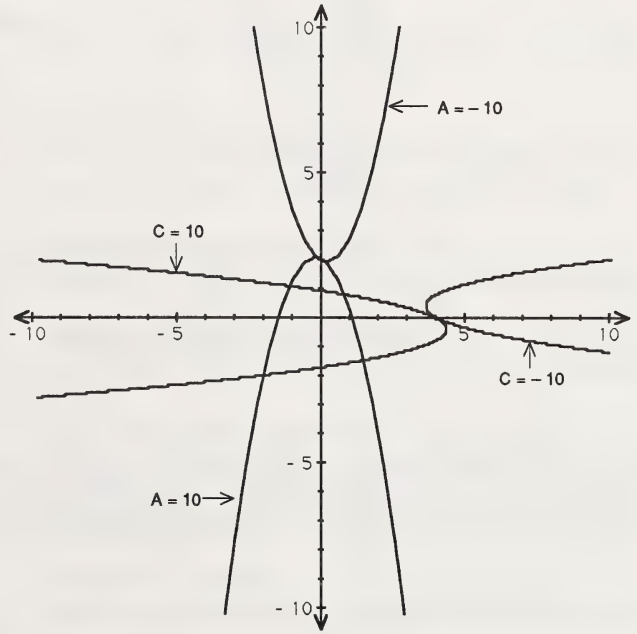
- $A = C$ gives a circle.
- Larger values of A and C shrink the circle until it disappears.
- Values of A, $C < 0$ result in no circle.

a. The resulting figure is a circle.

b. The circle disappears when A and C are very large. Increasing values of A and C produce a smaller circle.

c. No circle is produced when A and C are negative.

Investigation 4



Observations:

Four parabolas are produced when the following values for A and C are used.

- $A = 10$, $C = 0$ (opens down)
- $A = 0$, $C = 10$ (opens left)
- $A = -10$, $C = 0$ (opens up)
- $A = 0$, $C = -10$ (opens right)

a. The resulting shape when either $A = 0$ or $C = 0$ is a parabola.

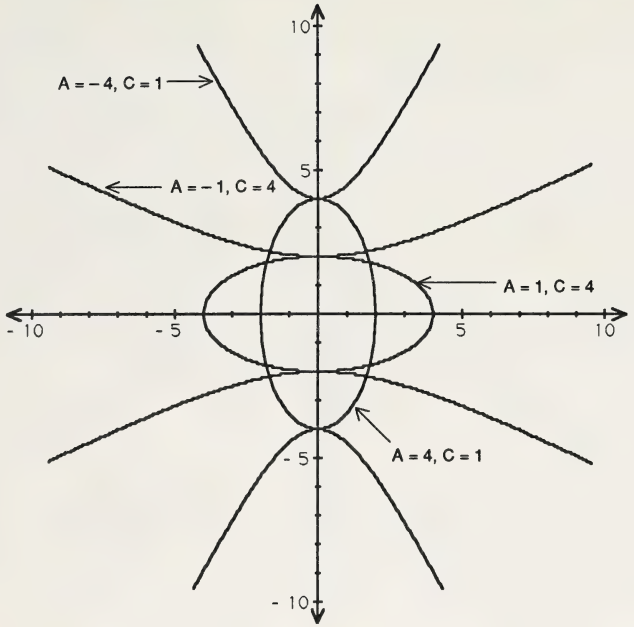
b. When $A > 0$, the parabola opens down.

c. When $A < 0$, the parabola opens up.

d. When $C > 0$, the parabola opens left.

e. When $C < 0$, the parabola opens right.

Investigation 5



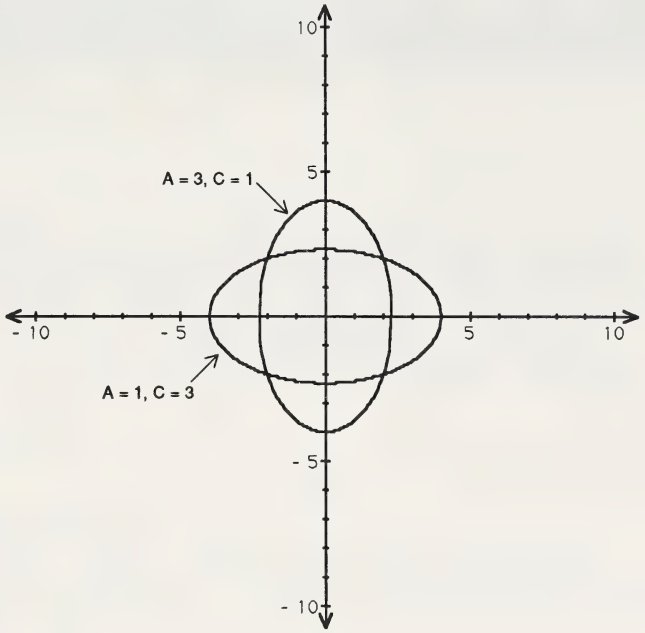
Observations:

- $A = 1,\ C = 4$ (ellipse with longer axis on x-axis)
- $A = 4,\ C = 1$ (ellipse with longer axis on y-axis)
- $A = -1,\ C = -4$ (no graph)
- $A = -4,\ C = 1$ (narrow hyperbola)
- $A = -1,\ C = 4$ (wider hyperbola)

a. The resulting shapes are ellipses when A and C are both positive. If A and C are both negative, no graph results.

b. The resulting shapes are hyperbolas when A and C have different signs.

Investigation 6



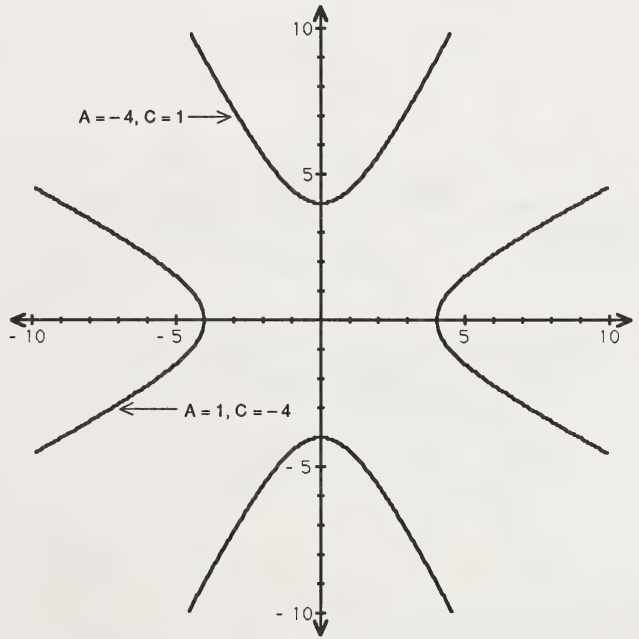
Observations:

- $A = 3$, $C = 1$ (ellipse with longer axis along y-axis)
- $A = 1$, $C = 3$ (ellipse with longer axis along x-axis)

Conclusion:

When A is greater than C, the ellipse has it longer axis along the y-axis. When A is less than C, the ellipse has its longer axis along the x-axis.

Investigation 7



Observations:

- $A = 1$, $C = -4$ (hyperbola on the x-axis)
- $A = -4$, $C = 1$ (hyperbola on the y-axis)

Conclusion:

When A is positive and C is negative, the hyperbola opens to the left and right. When A is negative and C is positive, the hyperbola opens up and down.

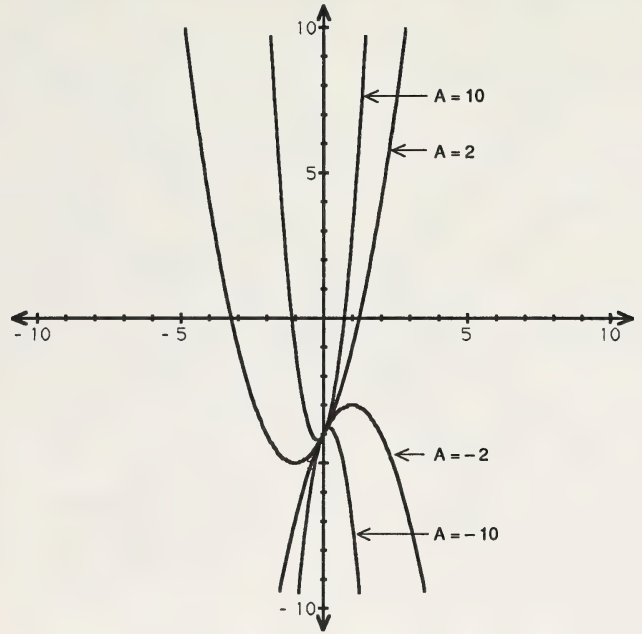
Summary:

When A equals C and F is constant, a circle results. Larger values of A and C result in smaller circles. When $A = 0$ or $C = 0$, the resulting curve is a parabola. When A and C are not equal but both are positive, an ellipse is the resulting curve. When A and C are not equal and one is positive while the other is negative, a hyperbola results.

Activity 3

Understand some of the effects of changing A or C one at a time in the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$.

Investigation 8



Observations:

- $A = -10$ (narrow graph, opens down)
- $A = -2$ (wider graph, opens down)
- $A = 2$ (wider graph, opens up)
- $A = 10$ (narrow graph, opens up)

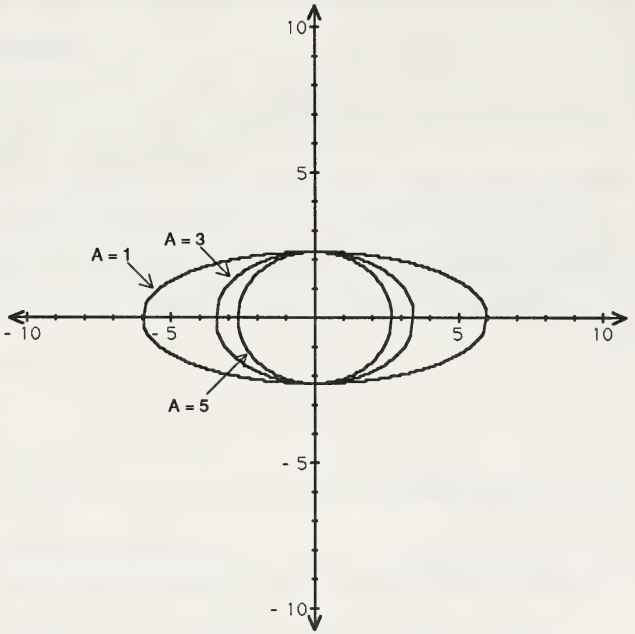
a. The graph opens down when $A < 0$ and up when $A > 0$.

b. The parabola is narrower when the absolute value of A is larger.

c. The vertex of the parabola shifts both horizontally and vertically as A changes.

d. Yes.

Investigation 9



Observations:

- $A = 1$ (long ellipse)
- $A = 3$ (shorter ellipse)
- $A = 5$ (shortest ellipse; almost like a circle)

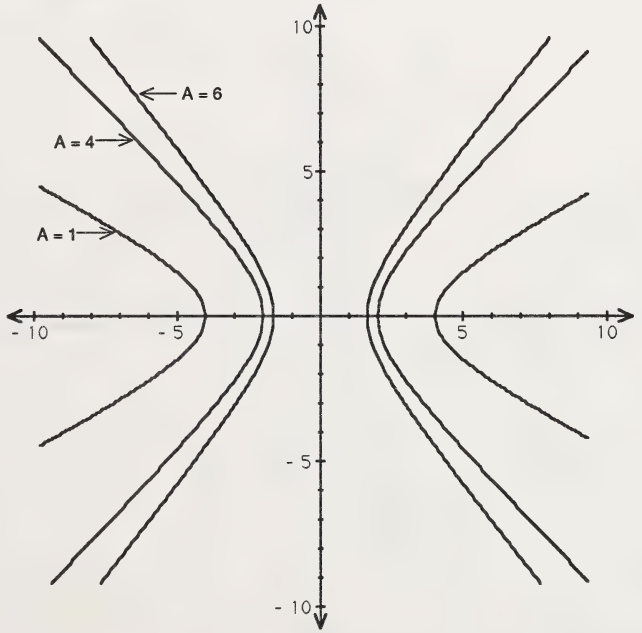
a. As the value of A increases from 1 to 3 to 5, the longer axis of the ellipse become shorter.

b. As A increases in size, the ellipse becomes more like a circle.

c. Letting $A = 7$ would result in the curve becoming a circle.

d. The ellipse will have its longer axis along the y-axis, and as the value of C increases from 1 to 3 to 5, the ellipse becomes more like a circle. At $C = 7$ the resulting curve will be a circle.

Investigation 10



Observations:

- $A = 1$ (narrow hyperbola)
- $A = 4$ (hyperbola becomes wider, arms closer)
- $A = 6$ (hyperbola becomes still wider, arms still closer together)

a. As the value of A increases from 1 to 4 to 6, the hyperbola becomes wider and the two parts move closer together.

b. The larger the value of A, the wider the hyperbola and the closer together the arms get.

c. The hyperbola is on the y-axis, but the effects of increasing the size of C are the same.

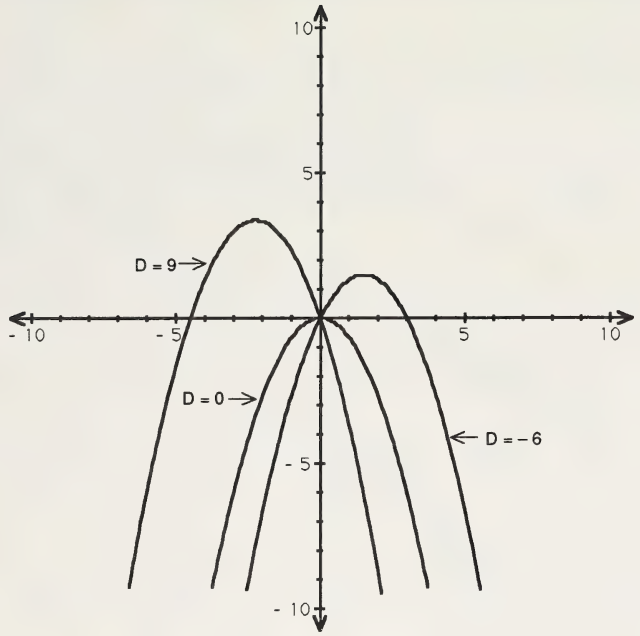
Summary:

For a parabola the larger the value of A or C, the narrower the parabola. For an ellipse the closer the values of A and C approach each other, the closer the ellipse comes to a circle. The greater the difference in relative sizes of A and C, the more elongated the ellipse. For the hyperbola, the larger the value of A or C, the flatter the arms of the hyperbola.

Activity 4

Understand some of the effects of making D or E equal to zero in $Ax^2 + Cy^2 + Dx + Ey + F = 0$.

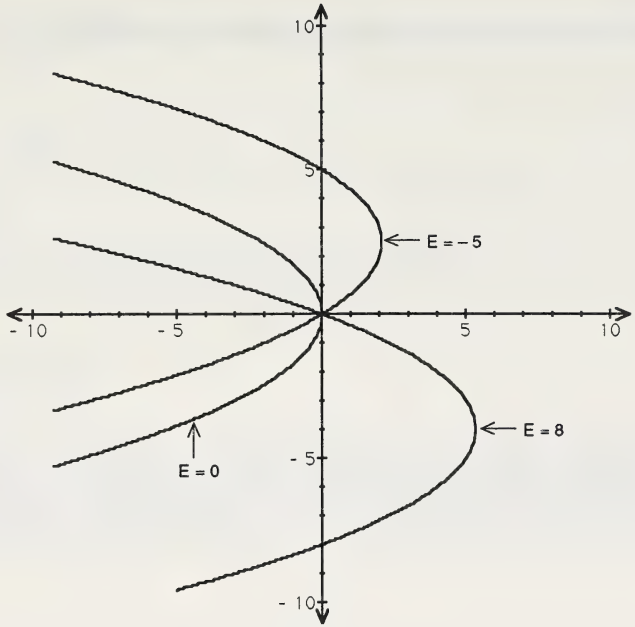
Investigation 11



Observations:

- $D = -6$ (vertex in quadrant 1)
- $D = 9$ (vertex in quadrant 2)
- $D = 0$ (vertex at the origin)

a. $D = 0$ puts the vertex of the parabola at the origin.



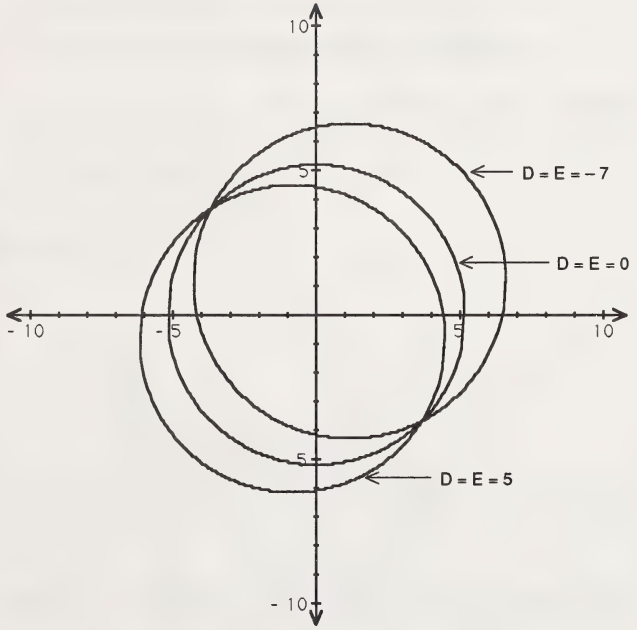
Observations:

- $E = -5$ (vertex in quadrant 1)
- $E = 8$ (vertex in quadrant 4)
- $E = 0$ (vertex at the origin)

b. $E = 0$ puts the vertex of the parabola at the origin.

Investigation 12

Circle

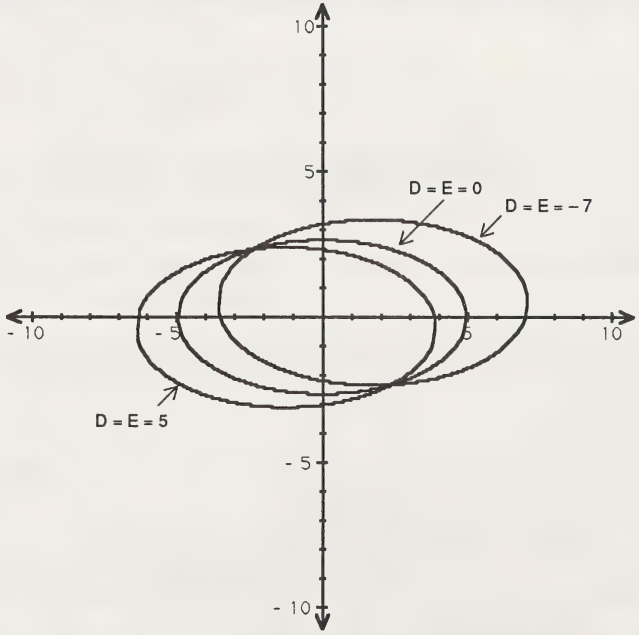


Observations:

- $D = E = -7$ (circle with centre in quadrant 1)
- $D = E = 5$ (circle with centre in quadrant 3)
- $D = E = 0$ (circle with centre at the origin)

a. $D = E = 0$ puts the circle with its centre at the origin.

Ellipse



Observations:

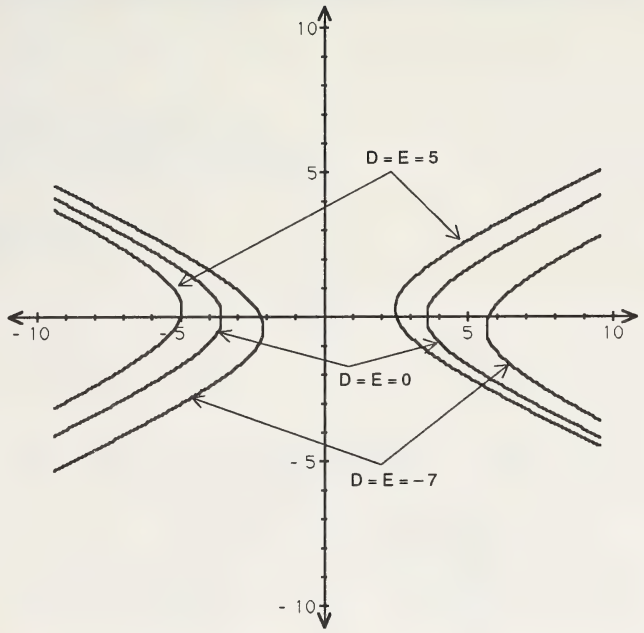
- $D = E = -7$ (ellipse with centre in quadrant 1)
- $D = E = 5$ (ellipse with centre in quadrant 3)
- $D = E = 0$ (ellispe with centre at the origin)

b. $D = E = 0$ puts the ellipse with its centre at the origin.

c. Yes.

d. Yes.

Hyperbola



Observations:

- $D = E = -7$ (moves the centre of the hyperbola right of the y-axis)
- $D = E = 5$ (moves the centre of the hyperbola left of the y-axis)
- $D = E = 0$ (puts the centre of the hyperbola at the origin)

e. $D = E = 0$

f. Yes.

g. Yes.

Summary:

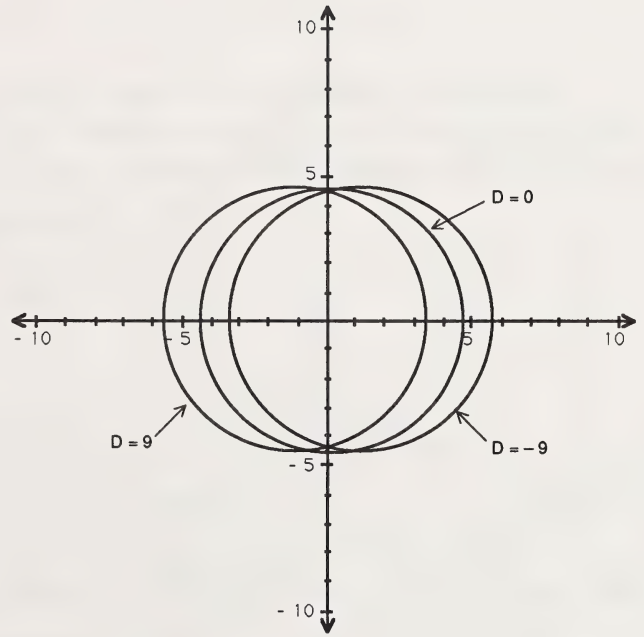
When D and E are zero, the centres of a circle, ellipse, and hyperbola are located at the origin.

Activity 5

Understand some of the effects of changing D, E, or F to various values in the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$.

Investigation 13

Circle

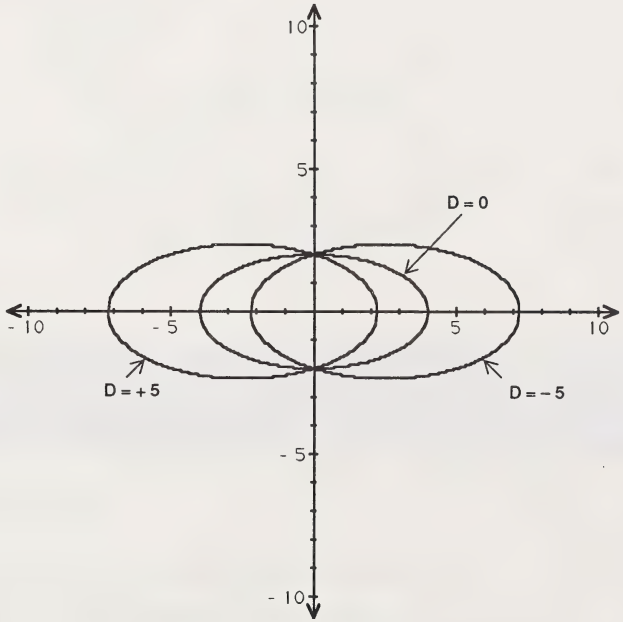


Observations:

- $D = -9$ (moves the circle right)
- $D = 9$ (moves the circle left)

a. The circle is moved left when D is positive.
b. The circle is moved right when D is negative.
c. Yes.

Ellipse

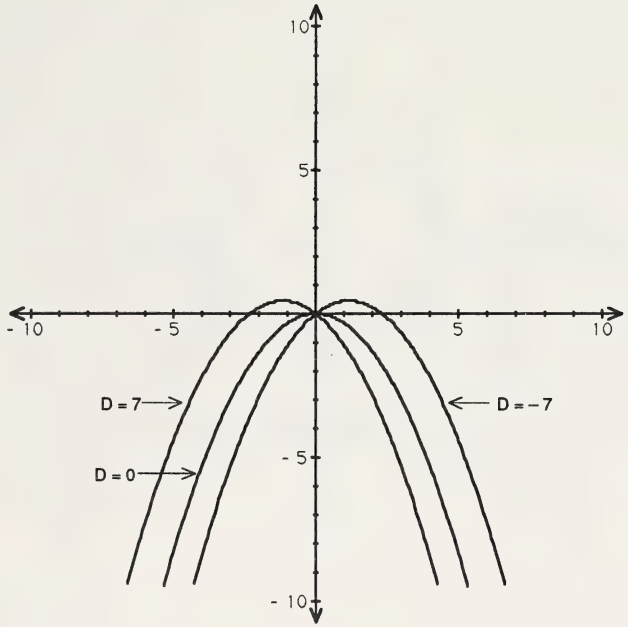


Observations:

- $D = 5$ (the ellipse moves left)
- $D = -5$ (the ellipse moves right)

d. The graph moves left when the value of D is positive.
e. The graph moves right when the value of D is negative.
f. Yes.

Parabola



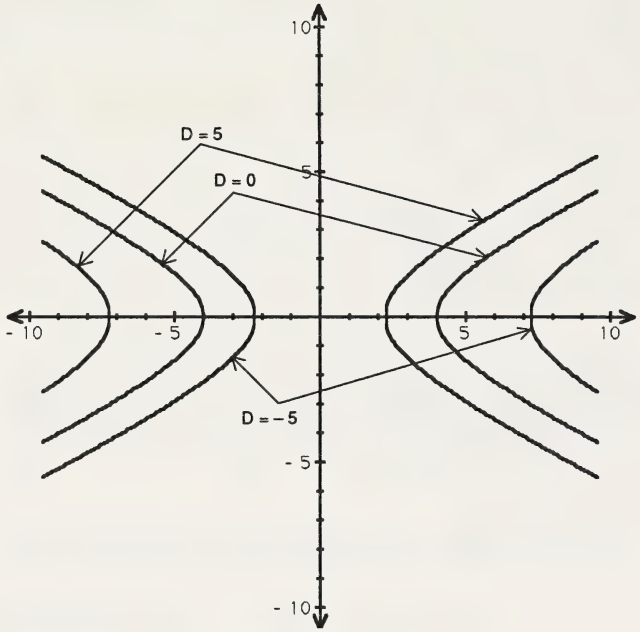
Observations:

- $D = 7$ (the graph moves left)
- $D = -7$ (the graph moves right)

g. When D is positive, the parabola moves left.
h. When D is negative, the parabola moves right.
i. Yes.

Hyperbola

$x^2 - 4y^2 - 16 = 0$

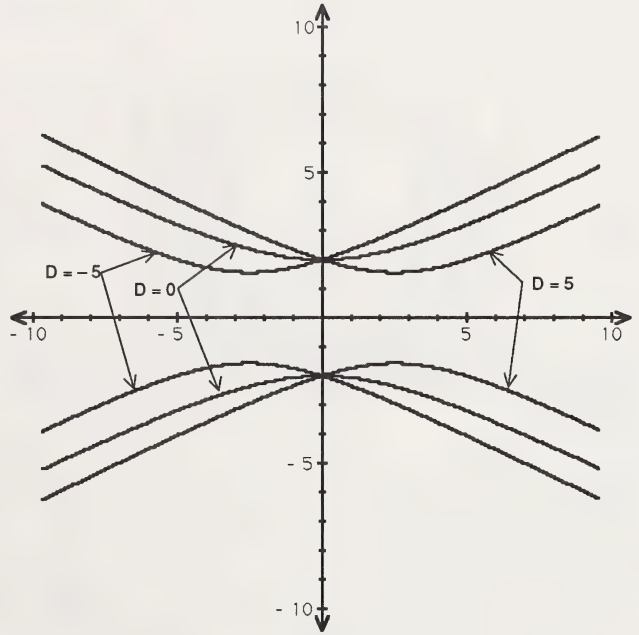


Observations:

- $D = 5$ (the hyperbola moves left)
- $D = -5$ (the hyperbola moves right)

j. When D is positive, the hyperbola moves left.
k. When D is negative, the hyperbola moves right.
l. Yes.

$-x^2+4y^2-16=0$



Observations:

- $D=5$ (hyperbola moves right)
- $D=-5$ (hyperbola moves left)

m. When D is positive, the hyperbola moves right.

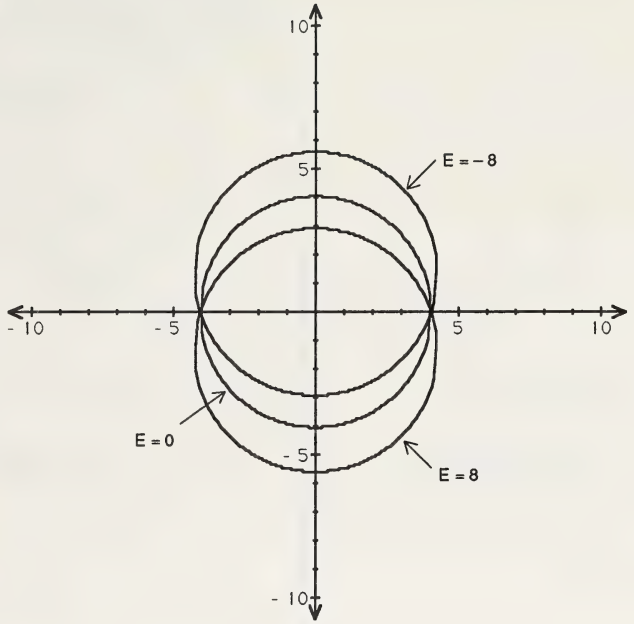
n. When D is negative, the hyperbola moves left.

o. No. The direction the hyperbola moves is opposite in this case. The sign of A and C affects the way D moves the graph.

p. Positive D moves the graph left, and negative D moves the graph right. The exceptions to this rule are in the parabola and hyperbola. When A has a negative value, then the effect of D is opposite – that is, positive D moves the graph right and negative D moves the graph left.

Investigation 14

Circle

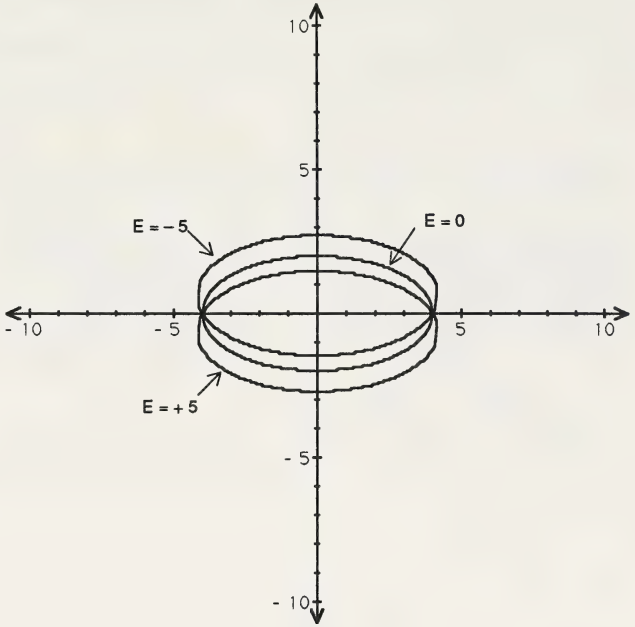


Observations:

- $E = 8$ (graph moves down)
- $E = -8$ (graph moves up)

a. When E is positive, the circle moves down.
b. When E is negative, the circle moves up.
c. Yes.

Ellipse

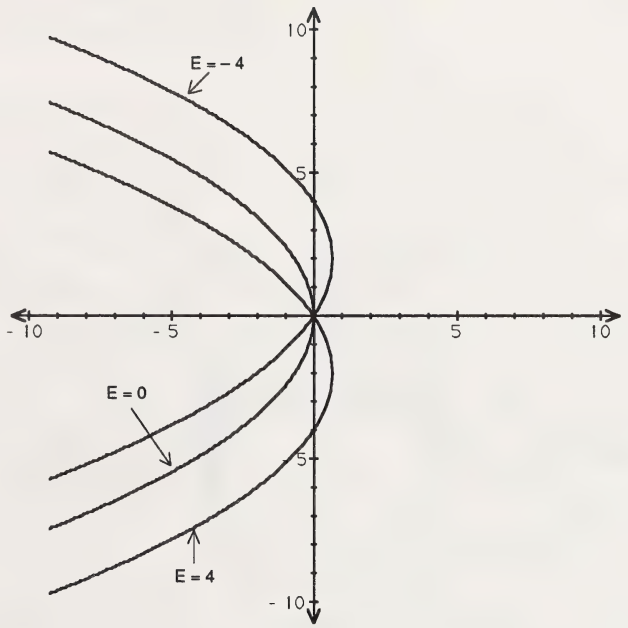


Observations:

- $E = 5$ (ellipse moves down)
- $E = -5$ (ellipse moves up)

d. When E is positive, the ellipse moves down.
e. When E is negative, the ellipse moves up.
f. Yes.

Parabola



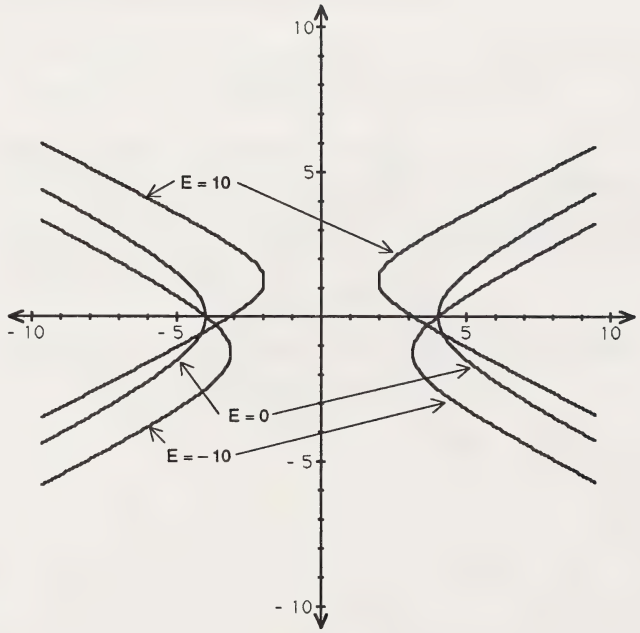
Observations:

- $E = 4$ (parabola moves down)
- $E = -4$ (parabola moves up)

g. The parabola moves down when E is positive.
h. The graph moves up when E is negative.
i. Yes.

Hyperbola

$x^2 - 4y^2 - 16 = 0$

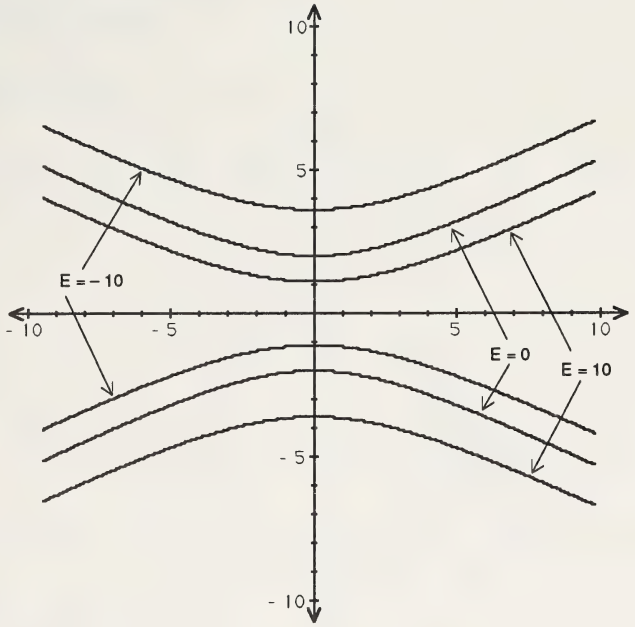


Observations:

- $E = 10$ (hyperbola moves up)
- $E = -10$ (hyperbola moves down)

j. The graph moves up when E is positive.
k. The graph moves down when E is negative.
l. Yes.

$-x^2+4y^2-16=0$



Observations:

- $E=10$ (hyperbola moves down)
- $E=-10$ (hyperbola moves up)

m. The graph moves down when E is positive.

n. The graph moves up when E is negative.

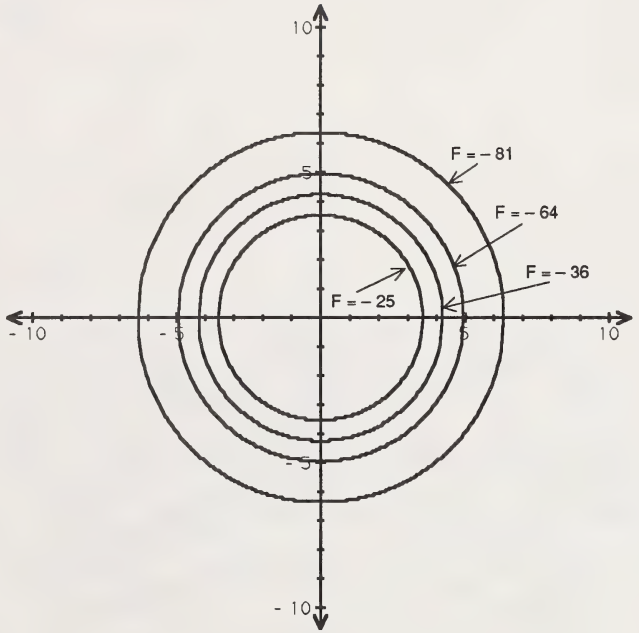
o. Yes.

p. The movement is in the opposite direction for each value of E.

q. Changing the value of E moves the graph up or down. Positive E values move the graph down for the circle and ellipse as well for the parabola and hyperbola when A is negative. Negative E values move these graphs up. When A is positive, the effects of E are opposite.

Investigation 15

Circle



Observations:

- $F = -25$ (circle)
- $F = -36$ (larger circle)
- $F = -64$ (larger circle)
- $F = -81$ (still larger circle)

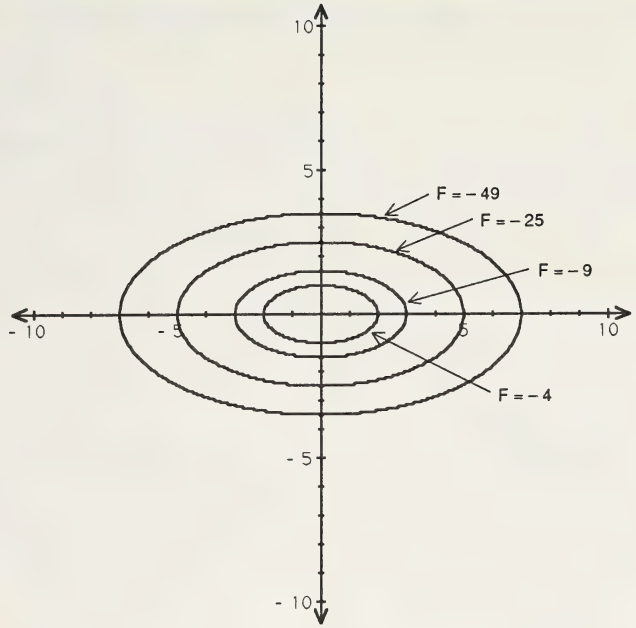
a. As the absolute value of F becomes larger, the circle becomes larger.

b. Yes, but the circle gets smaller.

c. No graph results when $F = 64$.

d. No two real numbers will give you a negative value when they are squared and added together.

Ellipse



Observations:

- $F = -4$ (small ellipse)
- $F = -9$ (larger ellipse)
- $F = -25$ (larger ellipse)
- $F = -49$ (still larger ellipse)

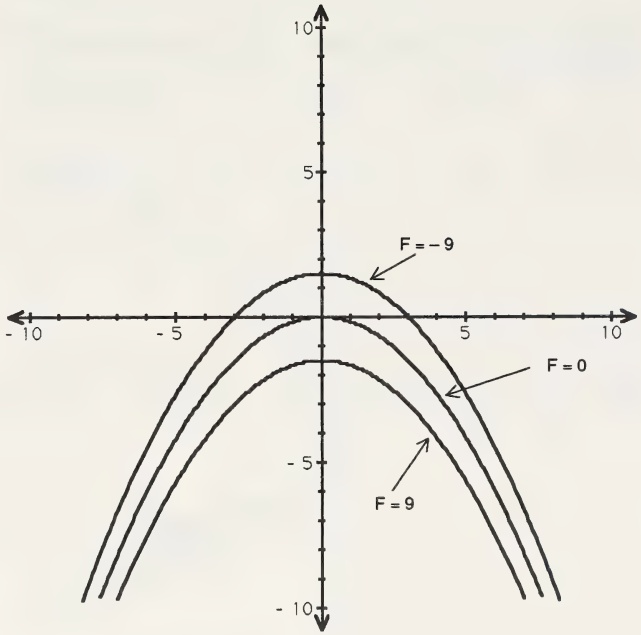
e. The larger the absolute value of F, the larger the ellipse.

f. Yes, except the graph is a vertical ellipse.

g. No. As in the case of the circle, you have two squared values equal to a negative value which is not possible.

Parabola

$x^2 + 6y = 0$



Observations:

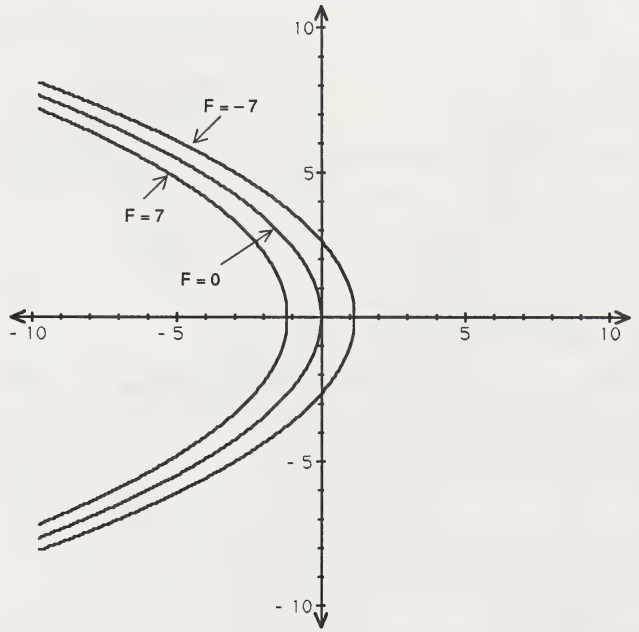
- $F = 9$ (parabola moves down)
- $F = -9$ (parabola moves up)

h. The parabola moves down when F is positive.

i. The parabola moves up when F is negative.

j. Yes.

$y^2 + 6x = 0$



Observations:

- $F = 7$ (parabola moves left)
- $F = -7$ (parabola moves right)

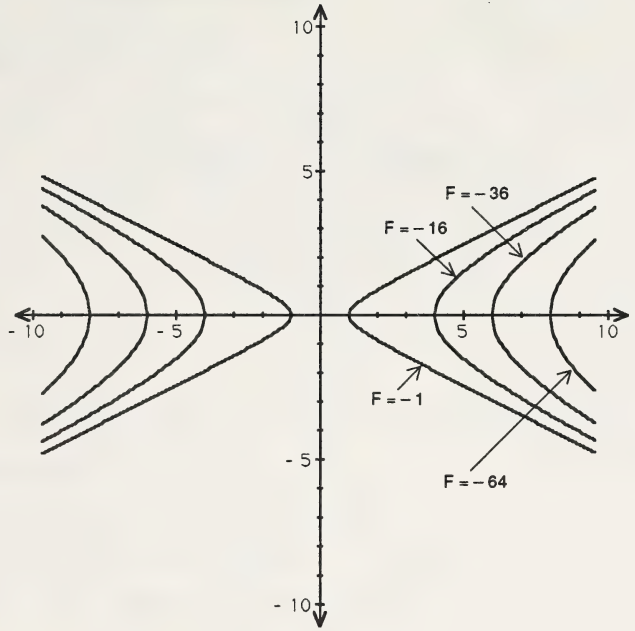
k. When F is positive, the parabola moves left.

l. When F is negative, the parabola moves right.

m. Yes.

Hyperbola

$x^2 - 4y^2 - 1 = 0$



Observations:

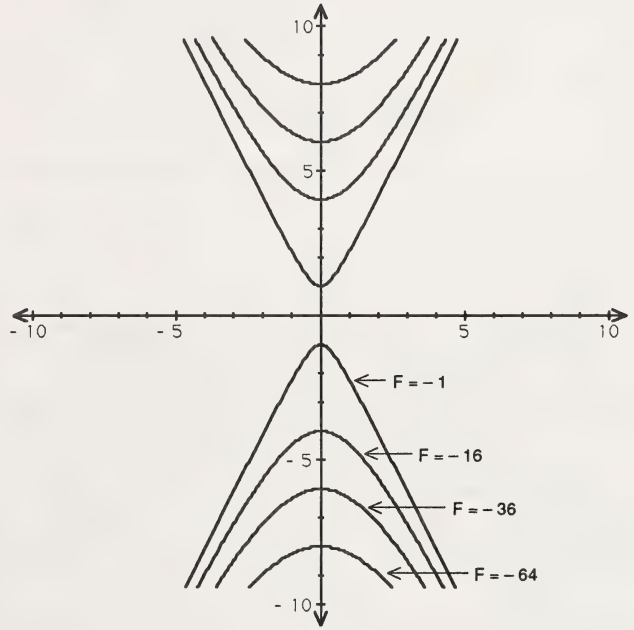
- $F = -1$ (hyperbola on x-axis)
- $F = -16$ (arms move away from y-axis)
- $F = -36$ (to arms spread apart more)
- $F = -64$ (arms are furthest apart)

n. As F is changed from -16 to -36 to -64, the two parts of the hyperbola spread apart and flatten out.

o. The graph rotates 90° to be on the y-axis. When the equation is multiplied by -1, it becomes the same equation as a hyperbola on the y-axis.

p. The arms spread apart and flatten out.

$-4x^2+y^2-1=0$



Observations:

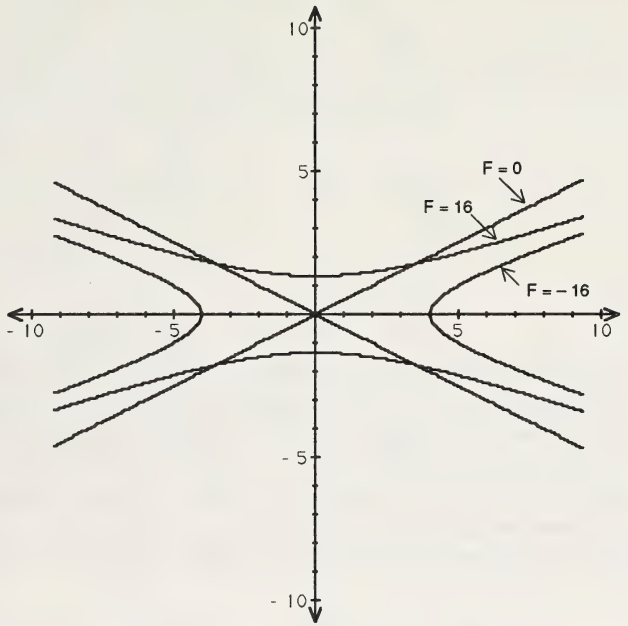
- $F=-1$ (hyperbola on the y-axis)
- $F=-16$ (arms move away from y-axis)
- $F=-36$ (arms spread apart more)
- $F=-64$ (arms are furthest apart)

q. The two parts of the hyperbola spread apart and the arms flatten out.

r. The graph rotates 90° to be on the x-axis. If you multiply the equation by -1, it becomes the same equation as a hyperbola on the x-axis.

s. The arms spread apart and flatten out.

$x^2 - 9y^2 + F = 0$



Observations:

- $F = 16$ (hyperbola on y-axis)
- $F = 0$ (two intersecting lines)
- $F = -16$ (hyperbola on y-axis)

t. The graph rotates 90° to the other axis.

u. The graph becomes two intersecting lines.

$$x^2 - 9y^2 + 0 = 0$$
$$(x - 3y)(x + 3y) = 0$$
$$x - 3y = 0 \text{ or } x + 3y = 0$$
$$y = \frac{1}{3}x \qquad y = -\frac{1}{3}x$$

$y = \frac{1}{3}x$ and $y = -\frac{1}{3}x$ are the equations of two intersecting lines.

v. This is the situation where the plane cuts the two cones on the axis of symmetry.

w. In the case of the circle and the ellipse, the larger the absolute value of F, the larger the curve. In the case of the parabola, if the parabola is on the y-axis, F moves the graph up or down. If the parabola is on the x-axis, F move the graph left or right. In the case of the hyperbola, F spreads the arms of the hyperbola apart and flattens the arms as the absolute value of F increases. A change from positive values, for F to negative values or vice versa, will switch the axis the hyperbola is situated on.

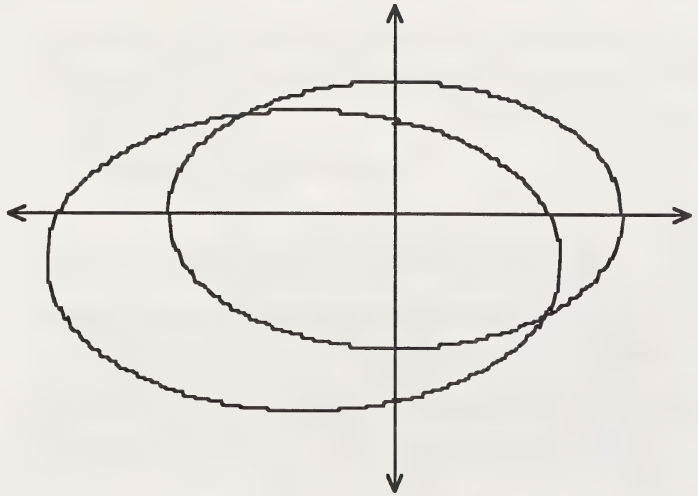
Summary:

In general D moves the graph left or right. In the case of the circle and ellipse, positive D moves the curves left and negative D moves the curves right. In the case of the parabola and hyperbola, D moves the curves left or right, but the direction the curve moves depends on the sign of A or C as well as the sign of D.

In general E moves the graph up or down. In the case of the circle and ellipse, positive E moves the curve down and negative E moves the curve up. In the case of the parabola and hyperbola, E moves the curves up or down, but again as with D the direction the curve moves depends on the sign of A or C as well as the sign of E.

The parameter F has different effects depending on the type of curve. The circle and ellipse are made larger or smaller. In the case of the parabola, changing F can move the parabola left, right, up, or down depending on the sign of A or C. With regards to the hyperbola the larger the absolute value for F, the flatter the arms become. A change from positive F to negative F switches the axis which the hyperbolas is on. These effects are the same for hyperbola on the y-axis and on the x-axis.

Investigation 16



These are possible equations. Other values for A, B, C, D, E, and F are possible.

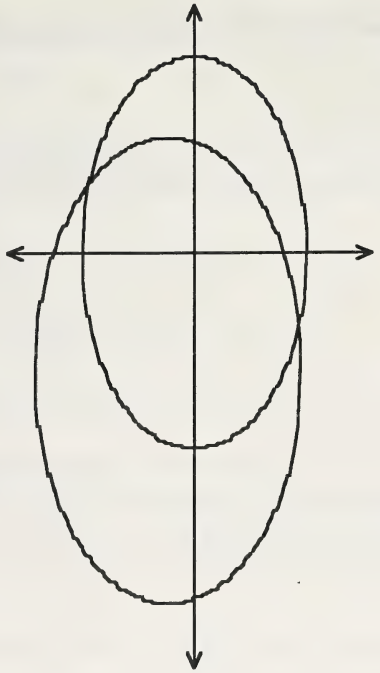
Observations:

- A possible equation for an ellipse with its centre at the origin is

  $2x^2+5y^2-25=0.$

- A possible equation for an ellipse with its centre in quadrant 3 is

  $2x^2+5y^2+6x+8y-25=0.$



Observations:

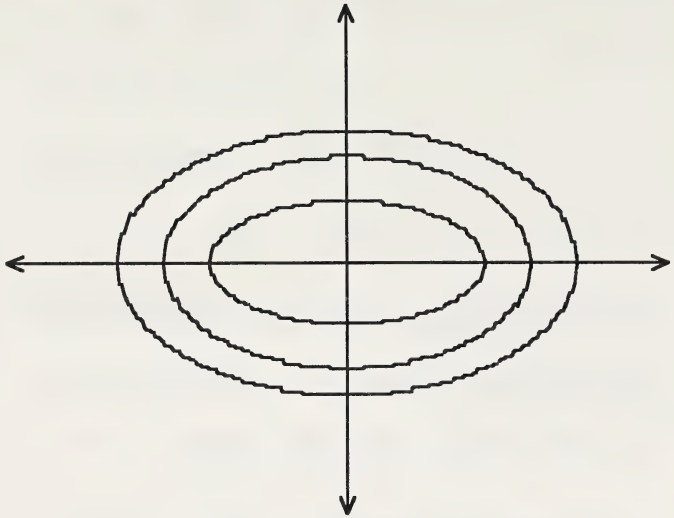
- A possible equation for an ellipse centred at the origin is

  $5x^2 + 2y^2 - 25 = 0$.

- A possible equation for an ellipse centred in quadrant 3 is

  $5x^2 + 2y^2 + 6x + 8y - 25 = 0$.

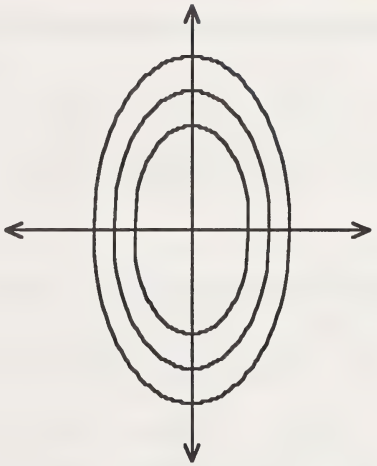
Investigation 17



Observations:

Starting from the inside ellipse going to the outside ellipse, possible equations for the ellipses include the following:

- $2x^2 + 5y^2 - 9 = 0$
- $2x^2 + 5y^2 - 16 = 0$
- $2x^2 + 5y^2 - 25 = 0$



Observations:

Starting from the inside ellipse going to the outside ellipse, possible equation for the ellipses are

- $5x^2+2y^2-9=0$
- $5x^2+2y^2-16=0$
- $5x^2+2y^2-25=0$

Investigation 18

These are possible values for the parameters and possible equations. Many others are possible.

a. $A=2 \quad B=6 \quad C=7 \quad D=0 \quad E=0 \quad F=-49$

Possible equation: $2x^2+6xy+7y^2-49=0$

- A and C are positive but not equal since the curve is an ellipse.
- B is not equal to zero since the ellipse is rotated.
- D and E are equal to zero since the ellipse is centred on the origin.

b. $A=-3 \quad B=0 \quad C=6 \quad D=0 \quad E=0 \quad F=-16$

Possible equation: $-3x^2+6y^2-16=0$

- Either A or C is negative, and A and C are not equal since curve is a hyperbola.
- A and F are negative since the curve is on the y-axis.
- If A and F are positive, this type of graph will also be given.
- D and E equal zero since the hyperbola is on the origin.

c. $A=4 \quad B=0 \quad C=4 \quad D=-6 \quad E=8 \quad F=-36$

Possible equation: $4x^2+4y^2-6x+8y-36=0$

- $A=C$ since the curve is a circle.
- $B=0$ since the curve is a circle.

- D is negative and E is positive since the centre is in quadrant 4.
- F is negative.
- All parameters could have the opposite sign, but this would really be the same equation.

d. $A=0 \quad B=0 \quad C=-7 \quad D=-3 \quad E=-5 \quad F=20$

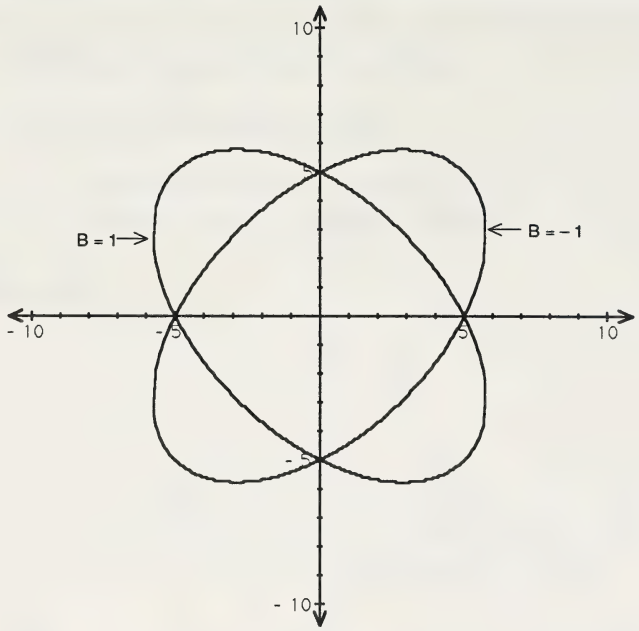
Possible equation: $-7y^2-3x-5y+20=0$

- $A=0$ since the graph is on the x-axis.
- $B=0$ since the curve is not rotated.
- D and E are negative since the vertex is in quadrant 4. (Remember that D has the opposite effect with a parabola when C is negative.)
- F is most likely positive since the vertex is to the right of the y-axis.

Activity 6

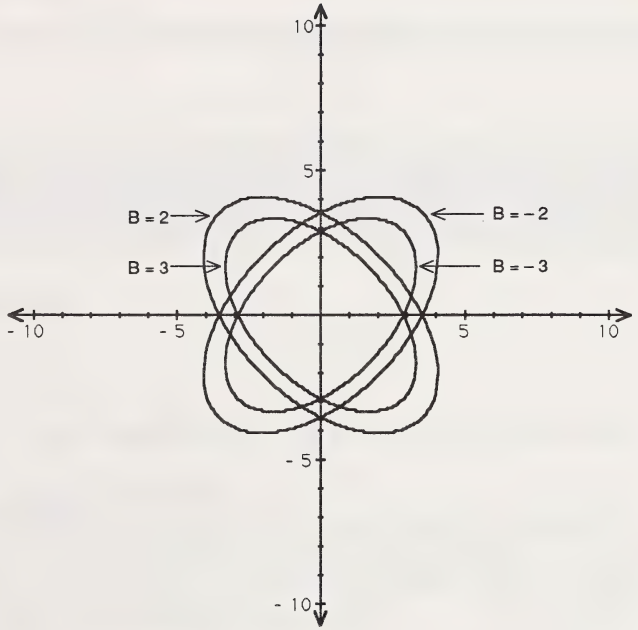
Understand some of the effects of changing B in $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$.

Investigation 19



Observations:

- $B=1$ (ellipse rotated with longer axis at 135°)
- $B=-1$ (ellipse rotated with longer axis at 45°)



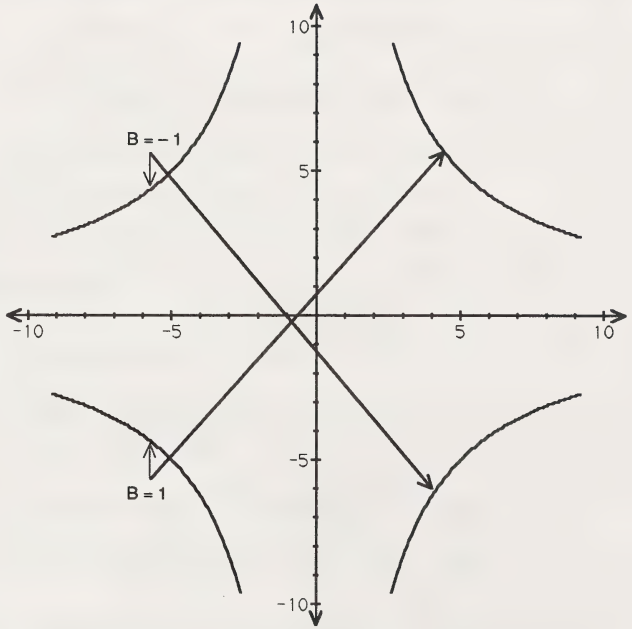
Observations:

- $B = -2, 2$ (ellipses rotated)
- $B = -3, 3$ (ellipses rotated and slightly smaller)

Conclusion:

B rotates the ellipse when the sign is changed and makes the ellipse smaller as the absolute value of B is increased.

Investigation 20



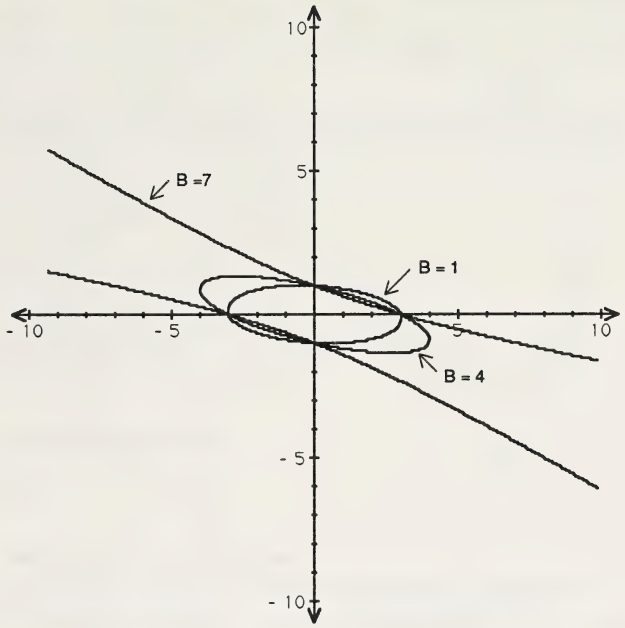
Observations:

- $B = 1$ (hyperbola with arms in quadrants 1 and 3)
- $B = -1$ (hyperbola with arms in quadrants 2 and 4)

a. The resulting curves are hyperbolas.

b. The position of the hyperbola is rotated 90°.

Investigation 21



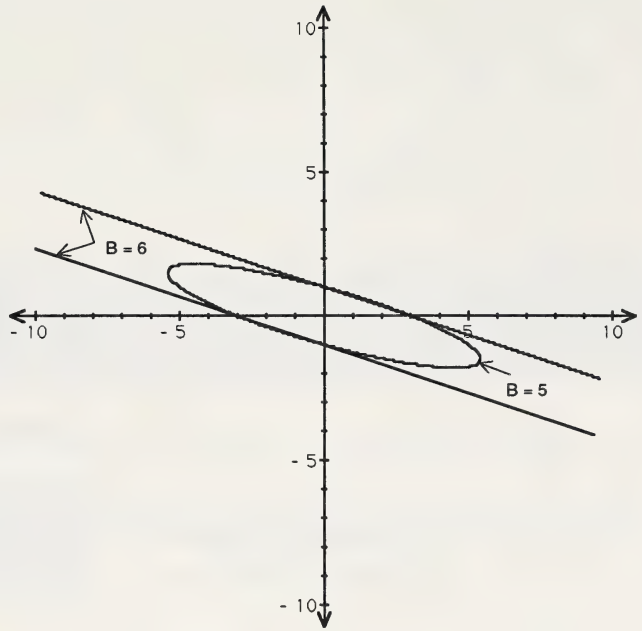
Observations:

- $B = 1$ (gives an ellipse on the x-axis slightly rotated)
- $B = 4$ (gives an ellipse rotated clockwise a little more)
- $B = 7$ (gives a rotated hyperbola)

a. You get an ellipse centred on the origin and slightly rotated.

b. You get an ellipse still centred on the origin and rotated slightly more.

c. When $B = 7$, the curve is a hyperbola.



Observations:

- $B = 5$ (graph is an ellipse again)
- $B = 6$ (graph is two parallel lines)

d. When $B = 5$, the graph is an ellipse again.

e. i. When $B = 6$, the graph becomes two parallel lines.

ii. When $B = 6$, the values of A, B, and C form a perfect-square relationship.

f. You get the same curves as with the positive values for B, except with negative values the rotation is counterclockwise as the absolute value of B increases.

g. Yes.

h. You get only one line.

$$x^2 + 6xy + 9y^2 = 0$$
$$(x + 3y)(x + 3y) = 0$$
$$x + 3y = 0 \text{ or } x + 3y = 0$$
$$y = -\frac{1}{3}x \qquad y = -\frac{1}{3}x$$

Although there are two equations, both represent the same line.

Summary:

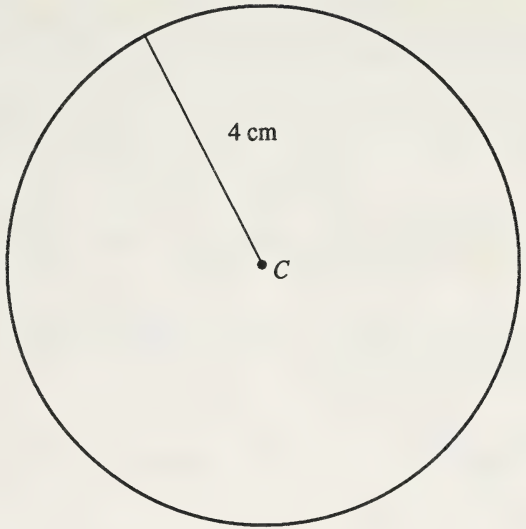
Changing the value of B changes the type of graph. In this case it was changed from an ellipse to two parallel lines to a hyperbola. Also, changing the value of B rotated the curves.

Exploring Topic 3

Activity 1

Recognize that each conic can be described as a locus of points.

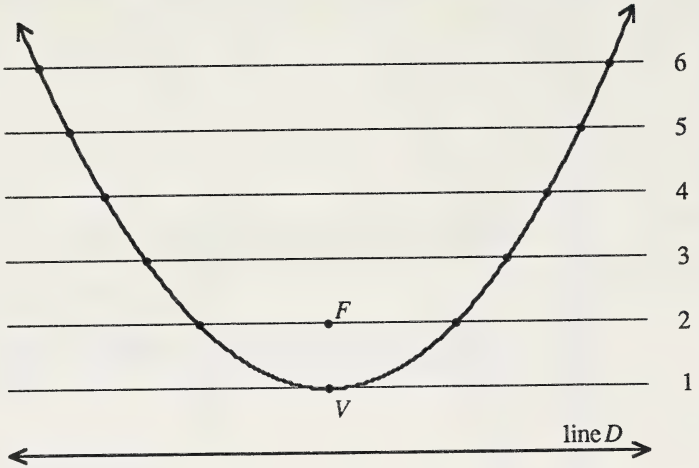
Investigation 1



Rule: All of the points are 4 cm from a fixed point C.

Locus definition: A **circle** is the locus of all points in a plane equidistant from a fixed point called the centre of the circle.

Investigation 2

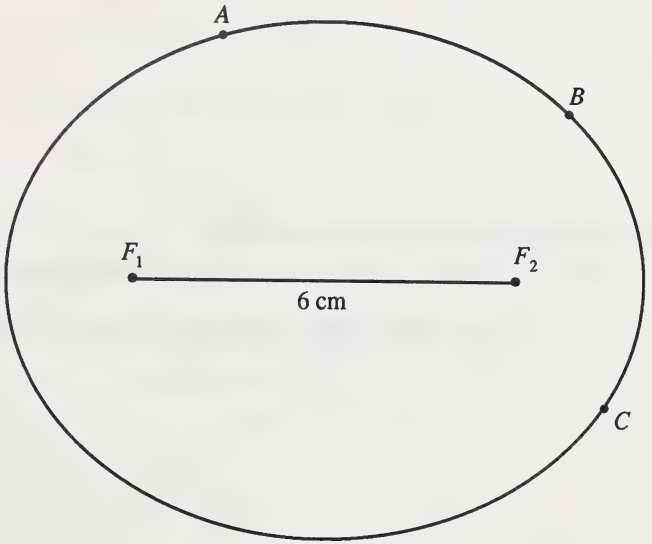


The curve is a parabola where $d(P_1F) = d(P_1D)$. (Note: P_1 represents any point on the parabola.)

Rule: Each point on the curve is equidistant from F and D.

Locus definition: A **parabola** is the locus of all points in a plane which are equidistant from a fixed point (focus) and a fixed line (directrix).

Investigation 3



a. The curve is an ellipse.

b. i. 10 cm
ii. 10 cm
iii. 10 cm

c. The sum of the distances from a point on the curve to the fixed points is 10 cm.

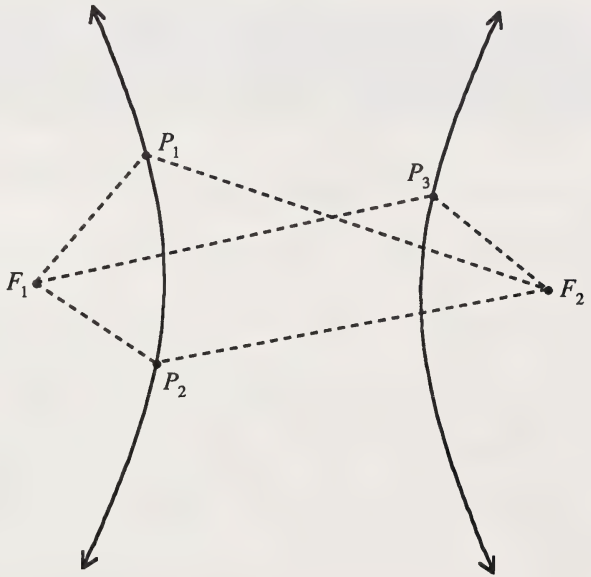
d. **Rule:** The sum of the distances from a point on the curve to the fixed points, F_1 and F_2, is a constant.

e. **Locus definition:** An **ellipse** is the locus of all points in a plane such that the sum of the distances from any point on the curve to two fixed points, called the foci, is constant.

f. If the string is shortened, the ellipse is smaller and more elongated in shape. If the string is longer, the ellipse is larger and assumes a more rounded shape.

g. As the holes get closer, the curve becomes more and more round. If $F_1 = F_2$, the ellipse changes to a circle.

Investigation 4



a. The conic is a hyperbola.

b. i. $|P_1F_1 - P_1F_2| = |3.4 \text{ cm} - 7.4 \text{ cm}| = 4 \text{ cm}$

ii. $|P_2F_1 - P_2F_2| = |2.5 \text{ cm} - 6.5 \text{ cm}| = 4 \text{ cm}$

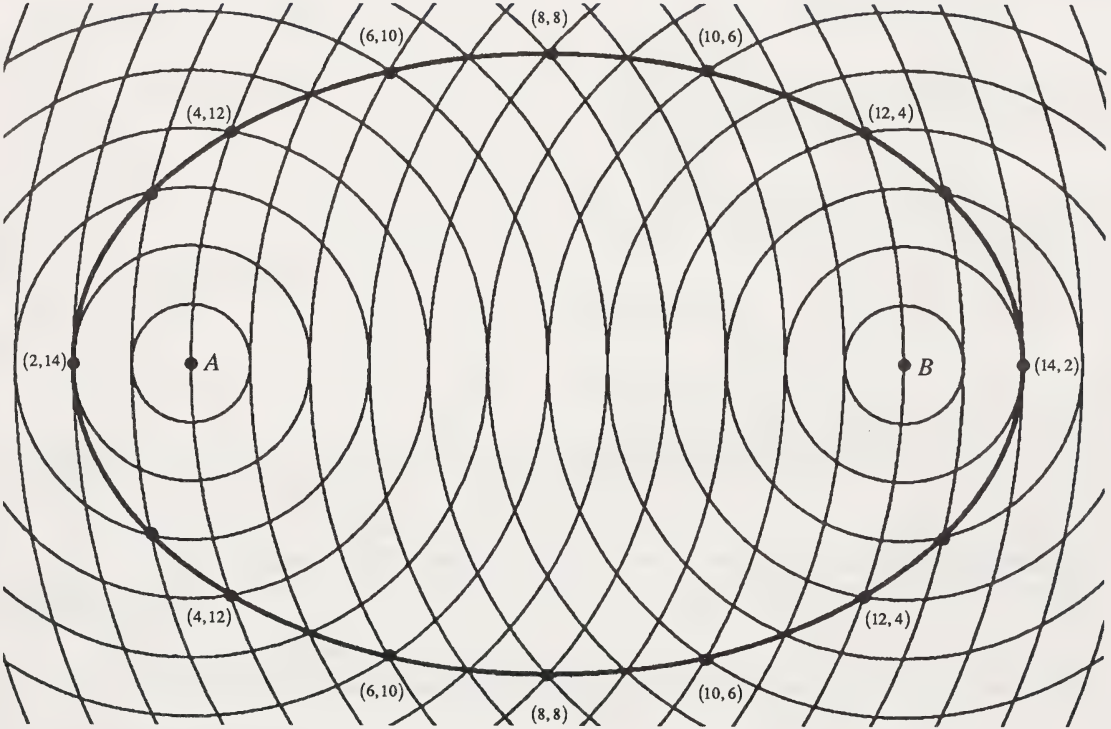
iii. $|P_3F_1 - P_3F_2| = |6.7 \text{ cm} - 2.7 \text{ cm}| = 4 \text{ cm}$

c. The measures are equal.

d. **Rule:** The absolute value of the difference of the distances from a point on the curve to the fixed points, F_1 and F_2, is a constant.

e. **Locus definition:** A **hyperbola** is the locus of all points in a plane such that the absolute value of the difference in distances from any point on the curve to two fixed points called the **foci** is constant.

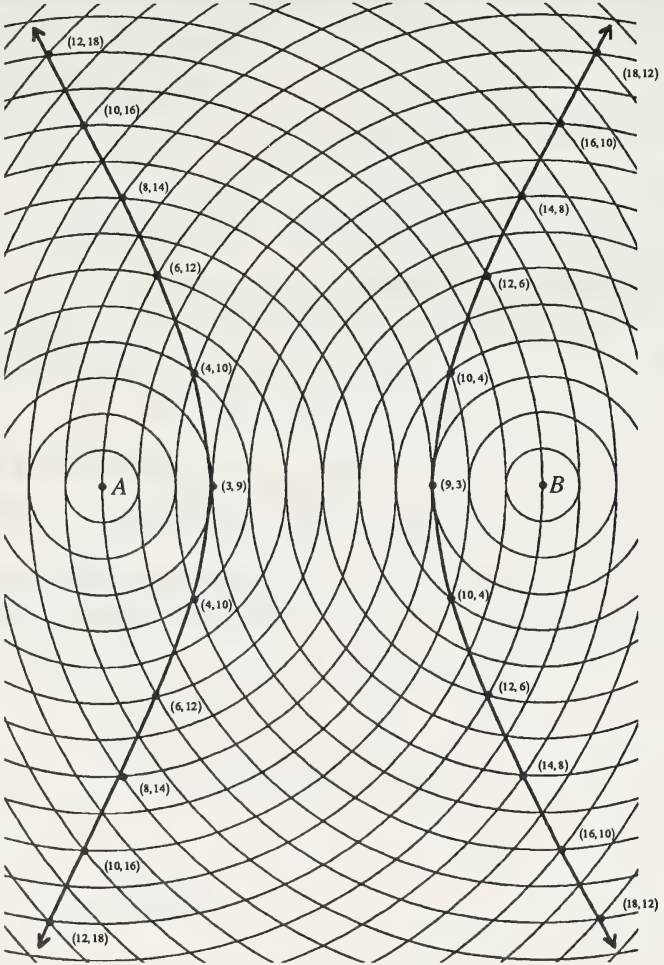
f. The graph is narrrower and the distance F_1F_2 must be greater than 4.



a. The curve is an ellipse.

b. This curve satisfies the locus definition for an ellipse.

Investigation 6



In the preceding graph the number 6 was used.

a. The curve is a hyperbola.

b. The curve satisfies the locus definition for a hyperbola.

Investigation 7

a. The value of z is -3.

b. The equation of the directrix is $y=-3$.

c. From the locus definition for a parabola $d(PF)=d(PD)$, and using the distance formula you can determine its equation.

Let $P(x, y)$ be any point on the curve.

The coordinates of F and D are $(0,3)$ and $(x,-3)$.

$$d(PF)=d(PD)$$

$$\sqrt{(x-0)^2+(y-3)^2}=\sqrt{(x-x)^2+[y-(-3)]^2}$$

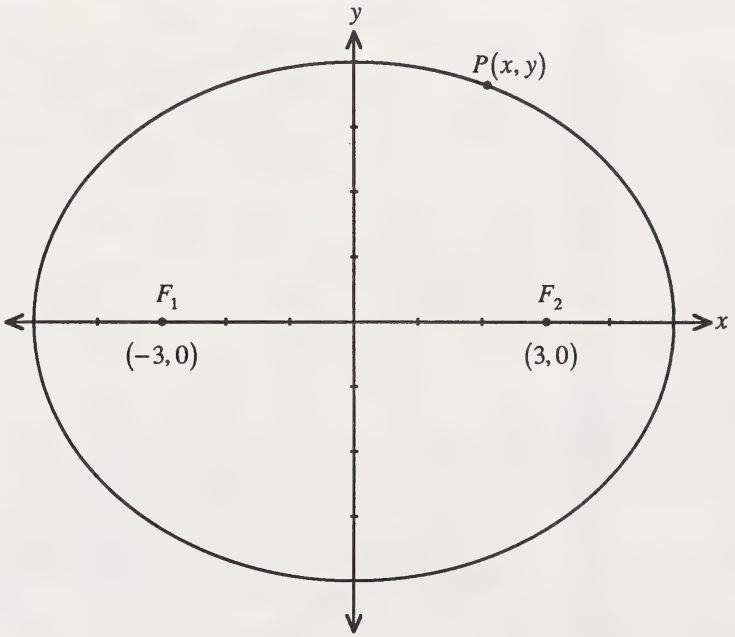
$$x^2+(y-3)^2=0+(y+3)^2$$

$$x^2+y^2-6y+9=y^2+6y+9$$

$$x^2-12y=0$$

The equation of the parabola is $x^2-12y=0$ where $A=1$, $B=0$, $C=0$, $D=0$, $E=-12$, and $F=0$.

Investigation 8



From the locus defintion for an ellipse $d(PF_1)+d(PF_2)$ is a constant where $F_1(-3,0)$ and $F_2(3,0)$ are foci. Using the distance formula you can determine its equation.

Let $P(x, y)$ be any point on the curve.

$$d(PF_1)+d(PF_2)=10$$

$$\sqrt{[x-(-3)]^2+(y-0)^2}+\sqrt{(x-3)^2+(y-0)^2}=10$$

$$\sqrt{(x+3)^2+y^2}+\sqrt{(x-3)^2+y^2}=10$$

$$\sqrt{(x+3)+y^2}=10-\sqrt{(x-3)^2+y^2}$$

$$\left(\sqrt{(x+3)^2+y^2}\right)^2=\left(10-\sqrt{(x-3)^2+y^2}\right)^2$$

$$(x+3)^2+y^2=100-20\sqrt{(x-3)^2+y^2}+(x-3)^2+y^2$$

$$x^2+6x+9+y^2=100-20\sqrt{(x-3)^2+y^2}+x^2-6x+9+y^2$$

$$12x-100=-20\sqrt{(x-3)^2+y^2}$$

$$(12x-100)^2=\left(-20\sqrt{(x-3)^2+y^2}\right)^2$$

$$144x^2-2400x+10\,000=400\left(x^2-6x+9+y^2\right)$$

$$144x^2-2400x+10\,000=400x^2-2400x+3600+400y^2$$

$$-256x^2-400y^2+6400=0 \qquad \text{(Divide each term by } -16.\text{)}$$

$$16x^2+25y^2-400=0$$

The equation of the ellipse is $16x^2+25y^2-400=0$.

Investigation 9

From the locus definition for a hyperbola $\left|d(PF_1)-d(PF_2)\right|$ is a constant, and using the distance formula you can determine its equation.

$P(x, y)$ is any point on the curve. The coordinates of F_1 and F_2 are $(-4, 0)$ and $(4, 0)$.

$$d(PF_1)-d(PF_2)=4$$

$$\sqrt{[x-(-4)]^2+(y-0)^2}-\sqrt{(x-4)^2+(y-0)^2}=4$$

$$\sqrt{(x+4)^2+y^2}=\sqrt{(x-4)^2+y^2}+4$$

$$\left(\sqrt{(x+4)^2+y^2}\right)^2=\left(\sqrt{(x-4)^2+y^2}+4\right)^2$$

$$(x+4)^2+y^2=(x-4)^2+y^2+8\sqrt{(x-4)^2+y^2}+16$$

$$x^2+8x+16+y^2=x^2-8x+16+y^2+8\sqrt{x^2-8x+16+y^2}+16$$

$$16x-16=8\sqrt{x^2-8x+16+y^2}$$

$$2x-2=\sqrt{x^2-8x+16+y^2}$$

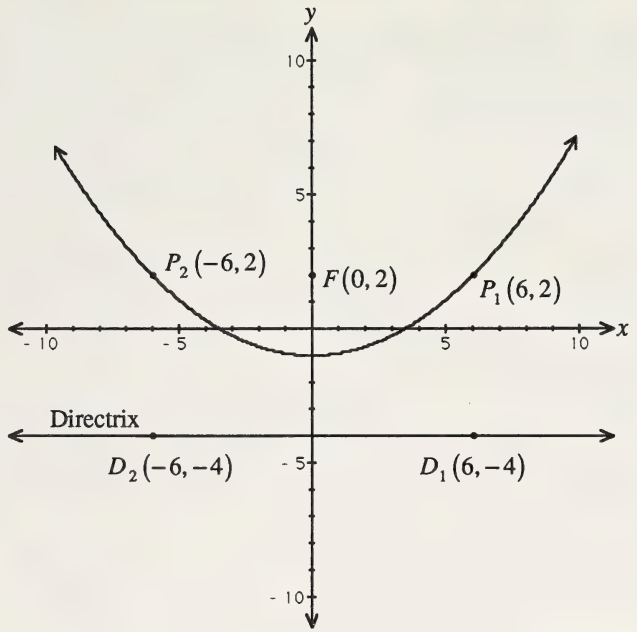
$$(2x-2)^2=\left(\sqrt{x^2-8x+16+y^2}\right)^2$$

$$4x^2-8x+4=x^2-8x+16+y^2$$

$$3x^2-y^2-12=0$$

The equation of the hyperbola is $3x^2-y^2-12=0$.

Investigation 10



Substitute a suitable value for y in the equation $x^2 - 12y - 12 = 0$, and solve for x.

Let $y = 2$.

$$x^2 - 12(2) - 12 = 0$$
$$x^2 = 36$$
$$x = \pm 6$$

The two points on the curve are $(-6, 2)$ and $(6, 2)$. By the locus defintion for a parabola $d(PF) = d(PD)$. Using the distance formula and the points $P_1(6,2)$, $F(0,2)$, and $D_1(6,-4)$, you can verify the locus definition for this parabola.

$$d(P_1F) = d(P_1D_1)$$
$$\sqrt{(6-0)^2 + (2-2)^2} = \sqrt{(6-6)^2 + [2-(-4)]^2}$$
$$\sqrt{(6)^2 + 0^2} = \sqrt{0^2 + (6)^2}$$
$$\sqrt{36} = \sqrt{36}$$
$$6 = 6$$

Now use the points $P_2(-6,2)$, $F(0,2)$, and $D_2(-6,-4)$ in the distance formula.

$$d(P_2F) = d(P_2D_2)$$
$$\sqrt{(-6-0)^2 + (2-2)^2} = \sqrt{[-6-(-6)]^2 + [2-(-4)]^2}$$
$$\sqrt{(-6)^2 + 0^2} = \sqrt{0^2 + (6)^2}$$
$$\sqrt{36} = \sqrt{36}$$
$$6 = 6$$

In both cases you can see that $d(PF) = d(PD)$.

1. a. circle
   b. arc
   c. straight line

2. The curve is a circle.

   Substitute the points $(-2, 3)$ and $(-4, 3)$ in the distance formula.

$$\begin{aligned} d &= \sqrt{[-2-(-4)]^2 + (3-3)^2} \\ &= \sqrt{(-2+4)^2 + 0^2} \\ &= \sqrt{2^2} \\ &= 2 \end{aligned}$$

   The point $(-4, 3)$ is 2 units from $(-2, 3)$.

   Therefore, $(-4, 3)$ lies on the locus.

3. The locus is a hyperbola.

$$d(PF_1) - d(PF_2) = \sqrt{(x-x_1)^2 + (y-y_1)^2} - \sqrt{(x-x_2)^2 + (y-y_2)^2}$$

   Let $(x, y) = (9, 6)$.

$$\begin{aligned} d(PF_1) - d(PF_2) &= \sqrt{[9-(-4)]^2 + (6-0)^2} - \sqrt{(9-4)^2 + (6-0)^2} \\ &= \sqrt{(13)^2 + (6)^2} - \sqrt{(5)^2 + (6)^2} \\ &= \sqrt{169+36} - \sqrt{25+36} \\ &= \sqrt{205} - \sqrt{61} \\ &= 6.5 \end{aligned}$$

   The constant of difference from $(9, 6)$ to the fixed points F_1 and F_2 is not equal to 6. Therefore, $(9, 6)$ does not lie on the curve.

4. Given $d(PF_1)+d(PF_2)=8$, substitute the points $(-2,3)$ and $(2,3)$ in the distance formula.

$$\sqrt{[x-(-2)]^2+(y-3)^2}+\sqrt{(x-2)^2+(y-3)^2}=8$$

$$\sqrt{(x+2)^2+(y-3)^2}=8-\sqrt{(x-2)^2+(y-3)^2} \quad \text{(Square both sides.)}$$

$$(x+2)^2+(y-3)^2=64-16\sqrt{(x-2)^2+(y-3)^2}+(x-2)^2+(y-3)^2$$

$$x^2+4x+4+y^2-6y+9=64-16\sqrt{(x-2)^2+(y-3)^2}+x^2-4x+4+y^2-6y+9$$

$$8x-64=-16\sqrt{(x-2)^2+(y-3)^2} \quad \text{(Divide by 8.)}$$

$$x-8=-2\sqrt{(x-2)^2+(y-3)^2}$$

$$(x-8)^2=\left(-2\sqrt{(x-2)^2+(y-3)^2}\right)^2$$

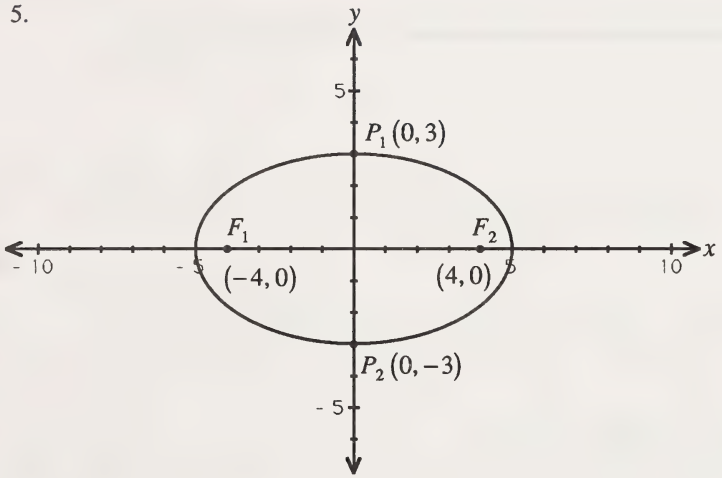
$$x^2-16x+64=4\left(x^2-4x+4+y^2-6y+9\right)$$

$$x^2-16x+64=4x^2-16x+16+4y^2-24y+36$$

$$3x^2+4y^2-24y-12=0$$

The equation of the ellipse is $3x^2+4y^2-24y-12=0$.

5.



Let $x=0$.

$$\therefore 25y^2-225=0$$
$$25y^2=225$$
$$y^2=9$$
$$y=\pm 3$$

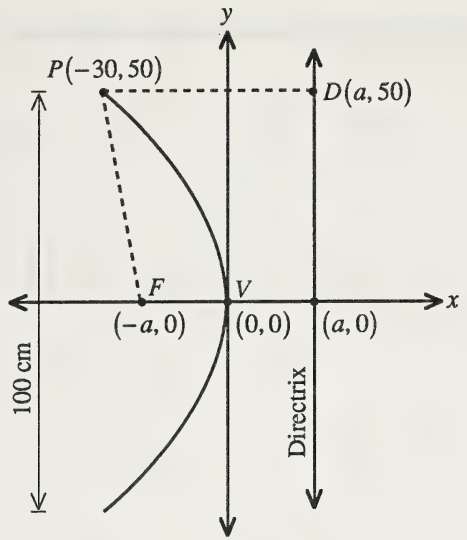
The two points are $P_1(0,3)$ and $P_2(0,-3)$.

$$k=d(P_1F_1)+d(P_1F_2)$$
$$=\sqrt{[0-(-4)]^2+(3-0)^2}+\sqrt{(0-4)^2+(3-0)^2}$$
$$=\sqrt{(4)^2+(3)^2}+\sqrt{(-4)^2+(3)^2}$$
$$=\sqrt{16+9}+\sqrt{16+9}$$
$$=5+5$$
$$=10$$

$$l=d(P_2F_1)+d(P_2F_2)$$
$$=\sqrt{[0-(-4)]^2+(-3-0)^2}+\sqrt{(0-4)^2+(-3-0)^2}$$
$$=\sqrt{(4)^2+(-3)^2}+\sqrt{(-4)^2+(-3)^2}$$
$$=\sqrt{16+9}+\sqrt{16+9}$$
$$=5+5$$
$$=10$$

Since $k=l$, you can state for this ellipse that $d(PF_1)+d(PF_2)=10$.

6.



$$d(PF) = d(PD)$$

$$\sqrt{[-30-(-a)]^2 + (50-0)^2} = \sqrt{(-30-a)^2 + (50-50)^2}$$

$$(-30+a)^2 + (50)^2 = (-30-a)^2 + 0^2$$

$$900 - 60a + a^2 + 2500 = 900 + 60a + a^2$$

$$120a = 2500$$

$$a = \frac{2500}{120}$$

$$a = 20.8$$

The distance from the vertex to the light source is 20.8 cm.

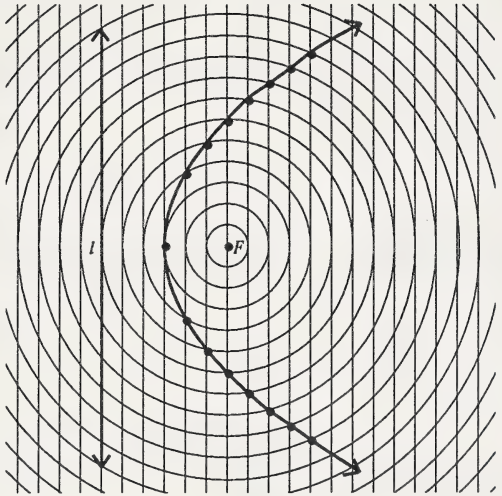
Exploring Topic 4

Activity 1

Describe conics as the locus of points such that the ratio of the distance between any point on the conic and a fixed point (focus) to the distance between the same point and a fixed line (directrix) is constant.

Investigation 1

These points determine a parabola.

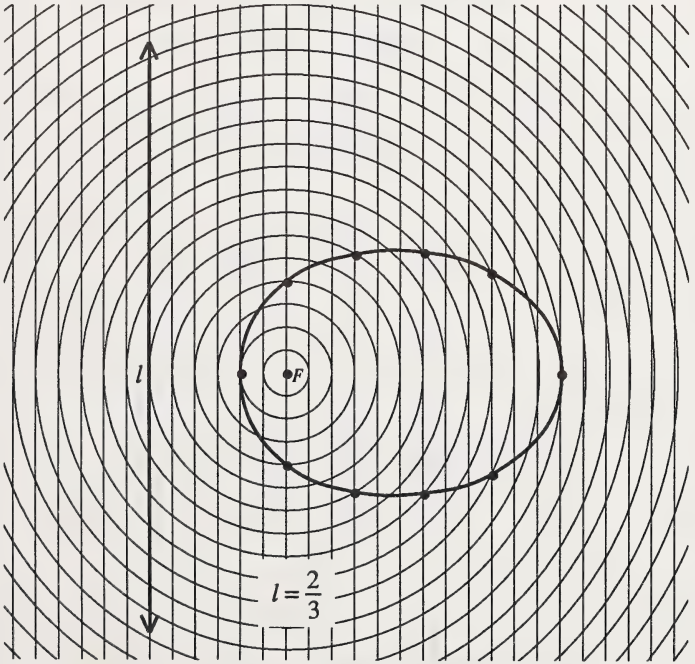


Investigation 2

Yes, you should get the same type of conic (parabola).

Investigation 3

The graph will vary. The following graph is just one example. The conic is an ellipse.



Investigation 4

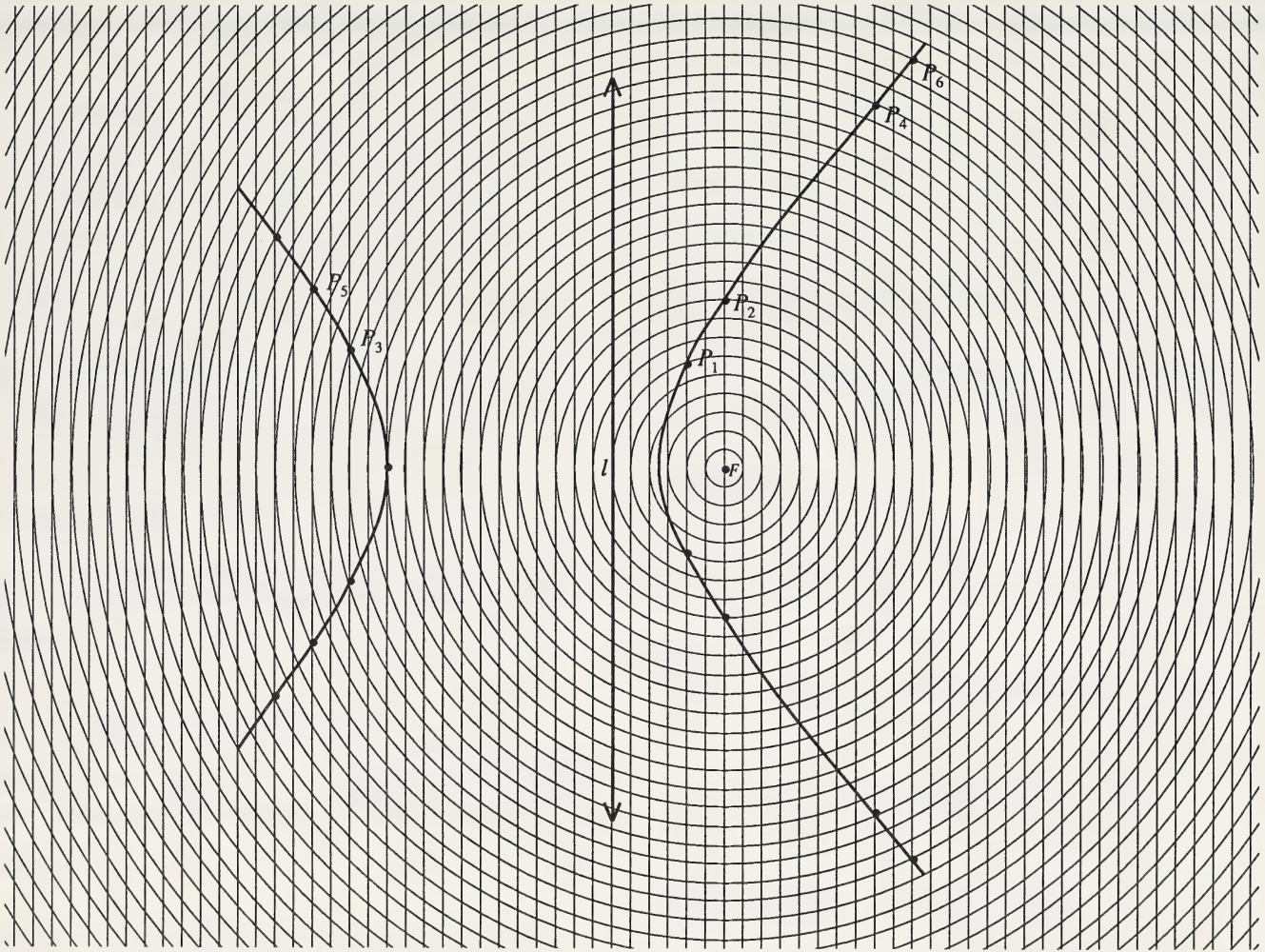
Graphs will vary. The larger eccentricity produces bigger but flatter ellipses. When e is close to zero, the ellipse is close to a circle. If $0 < e < 1$, then the conic is an ellipse.

Investigation 5

Graphs will vary. The ellipses are similar in shape. If the distance between the directrix and focus is larger, then the ellipse will be larger in size.

Investigation 6

The following graph is a hyperbola.


A
P_6
P_4
P_5
P_3
P_2
P_1
l
F

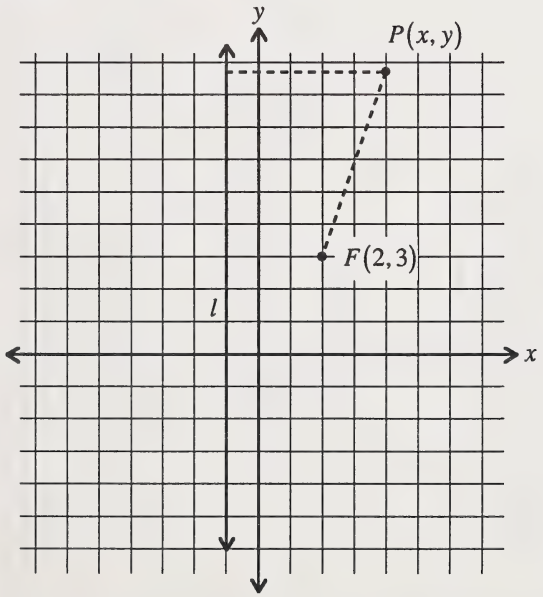
Investigation 7

As the eccentricity gets larger, the hyperbola becomes flatter.

Investigation 8

If you move the directrix to the right, the hyperbola will be moved to the right. If you move the directrix to the left, the hyperbola will be moved to the left. Although the focus and eccentricity would remain the same, the hyperbola would not be the same.

Investigation 9



$$e = \frac{\text{distance from } P \text{ to } F}{\text{distance from } P \text{ to } l}$$

$$\frac{3}{2} = \frac{\sqrt{(x-2)^2 + (y-3)^2}}{x+1}$$

$$3(x+1) = 2\sqrt{(x-2)^2 + (y-3)^2}$$

$$9(x+1)^2 = 4\left[(x-2)^2 + (y-3)^2\right]$$

$$9x^2 + 18x + 9 = 4\left(x^2 - 4x + y^2 - 6y + 13\right)$$

$$9x^2 + 18x + 9 = 4x^2 - 16x + 4y^2 - 24y + 52$$

$$5x^2 - 4y^2 + 34x + 24y - 43 = 0$$

This conic is a hyperbola.

Investigation 10

$$1 = \frac{\sqrt{(x-2)^2 + (y-3)^2}}{x+1}$$

$$(x+1)^2 = (x-2)^2 + (y-3)^2$$

$$x^2 + 2x + 1 = x^2 - 4x + 4 + y^2 - 6y + 9$$

$$y^2 - 6x - 6y + 12 = 0$$

This is a parabola.

$$\frac{3}{4}=\frac{\sqrt{(x-2)^2+(y-3)^2}}{x+1}$$

$$9(x+1)^2=16\left[(x-2)^2+(y-3)^2\right]$$

$$9x^2+18x+9=16x^2-64x+64+16y^2-96y+144$$

$$7x^2+16y^2-82x-96y+199=0$$

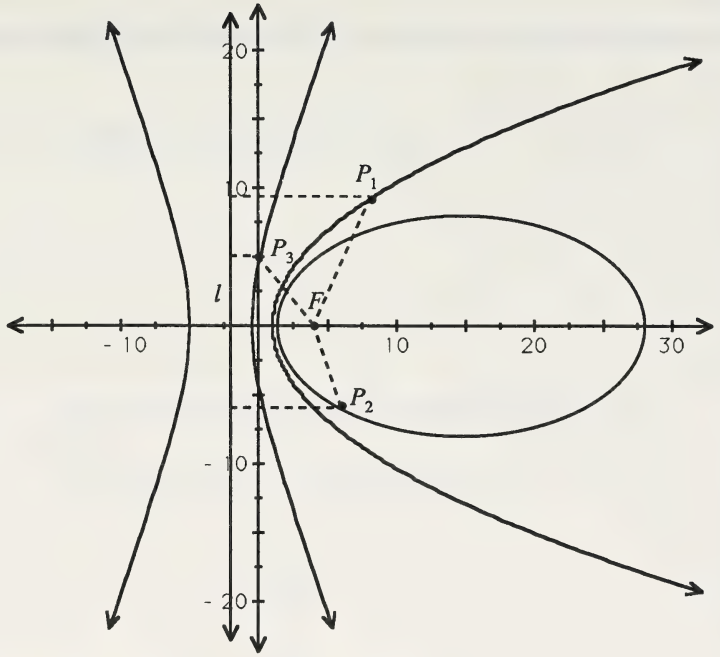
This is an ellipse.

Definitions:

If $e=1$, the conic is a parabola.
If $0<e<1$, the conic is a ellipse.
If $e>1$, the conic is a hyperbola.

Investigation 11

The following diagram shows one possible way to get the answer. You may want to try a different point. The answer should be the same.



$$\frac{d(P_1F)}{d(P_1l)}=\frac{2.1\text{ cm}}{2.1\text{ cm}}=1 \quad \text{(for the parabola)}$$

$$\frac{d(P_2F)}{d(P_2l)}=\frac{1.2\text{ cm}}{1.6\text{ cm}}=\frac{3}{4} \quad \text{(for the ellipse)}$$

$$\frac{d(P_3F)}{d(P_3l)}=\frac{1.2\text{ cm}}{0.4\text{ cm}}=3 \quad \text{(for the hyperbola)}$$

The eccentricity of the parabola is 1.

The eccentricity of the ellipse is $\frac{3}{4}$.

The eccentricity of the hyperbola is 3.

1. The eccentricity does not affect the shape of a parabola. If the focus is closer to the directrix, the parabola would be narrower and its vertex would be closer to the directrix.

2. If the directrix and focus remain the same, then a larger ellipse would have a larger eccentricity.

3. If the directrix and focus remain the same, the hyperbola with a smaller eccentricity would be narrower and the two sections of the hyperbola would be farther apart.

4. $$\frac{\sqrt{(x-5)^2+(y-2)^2}}{x-3}=\frac{4}{5}$$

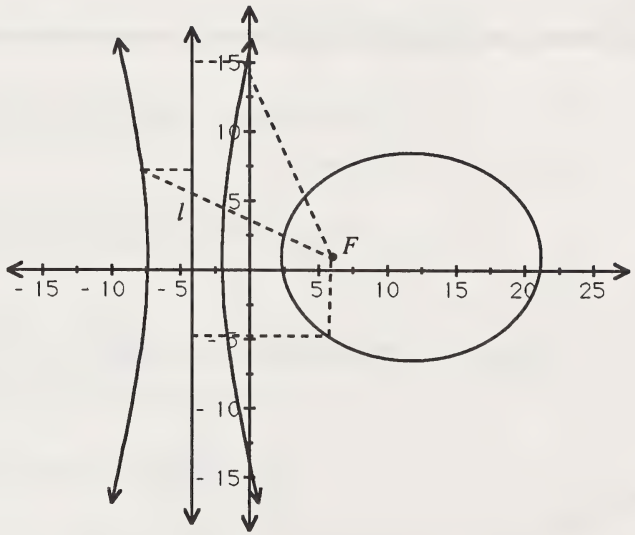
$$25\left(x^2-10x+y^2-4y+29\right)=16(x-3)^2$$

$$25x^2-250x+25y^2-100y+725=16x^2-96x+144$$

$$9x^2+25y^2-154x-100y+581=0$$

This is an ellipse.

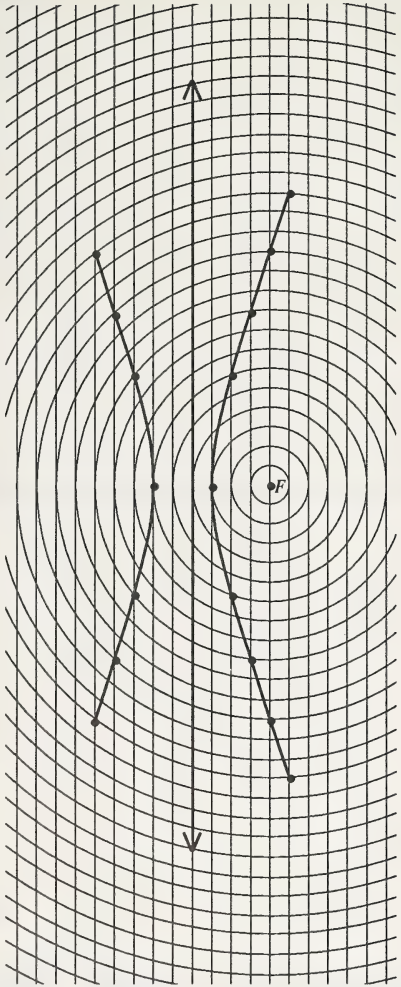
5.



The eccentricity of the ellipse is $\frac{1.20\text{ cm}}{2.05\text{ cm}} \doteq 0.59$.

The eccentricity of the hyperbola is $\frac{3.20\text{ cm}}{0.85\text{ cm}} \doteq 3.76$.

6. a.



b. The conic is a hyperbola.

7. $$\frac{\text{Distance between } P \text{ and the focus}}{\text{Distance between } P \text{ and the directrix}} = \frac{\sqrt{(4-5)^2+(0-0)^2}}{4-\frac{16}{5}}$$

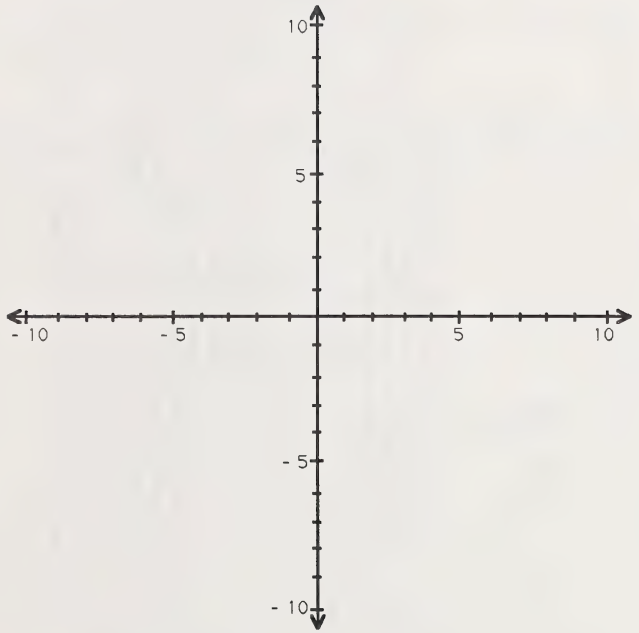
$$= \frac{1}{\left(\frac{4}{5}\right)}$$

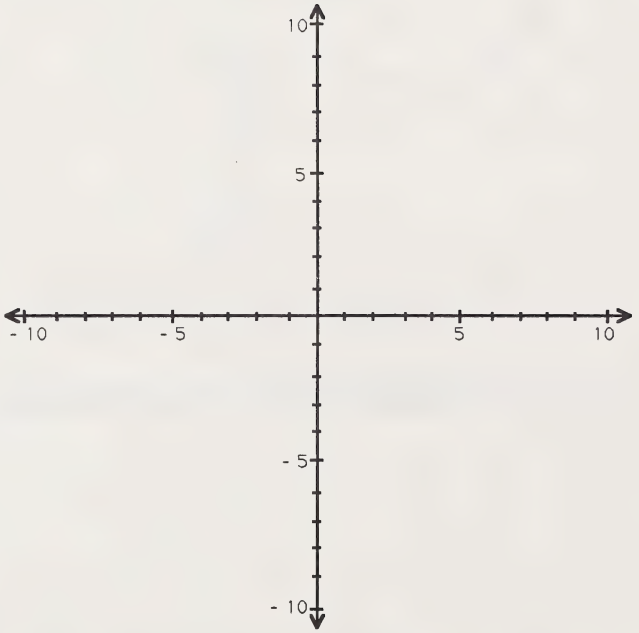
$$= \frac{5}{4}$$

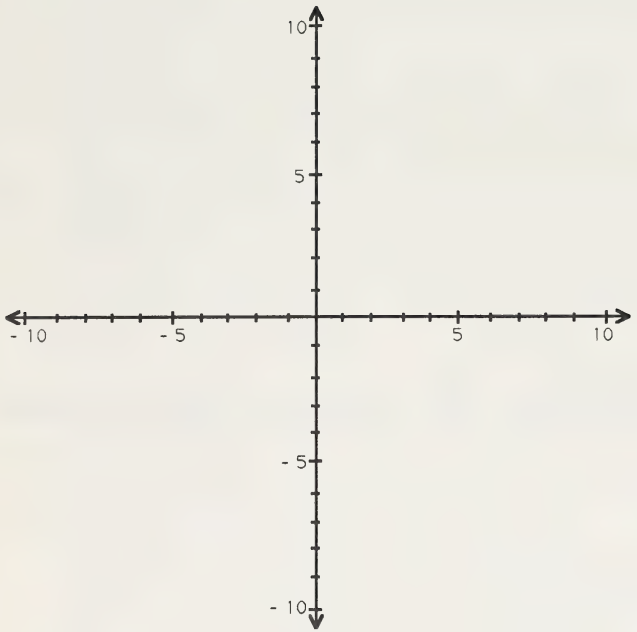
Since the eccentricity of the conic is $\frac{5}{4}$, point $(4, 0)$ is on the conic.

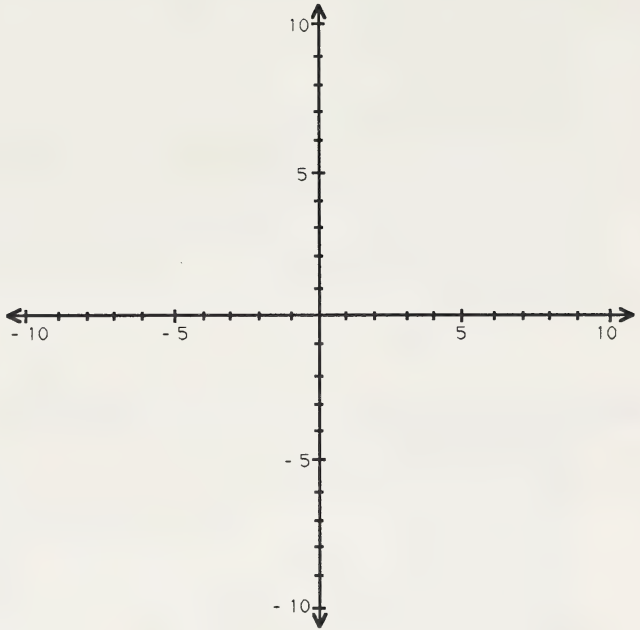
Appendix B
Graphing Materials

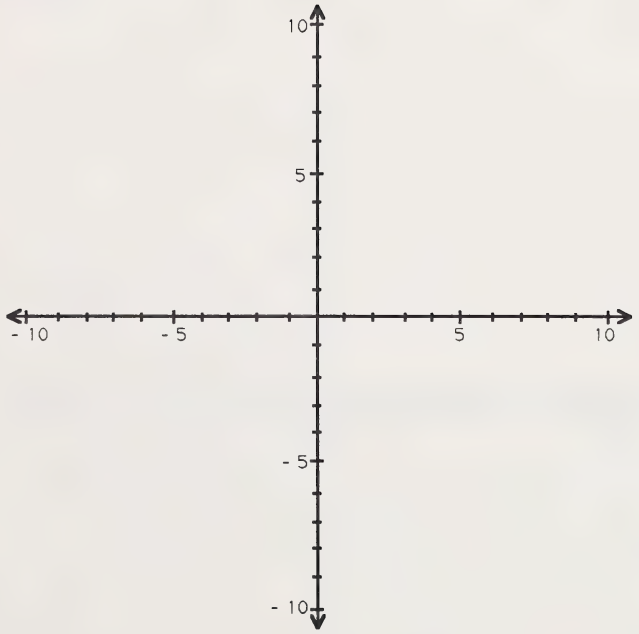
Grids

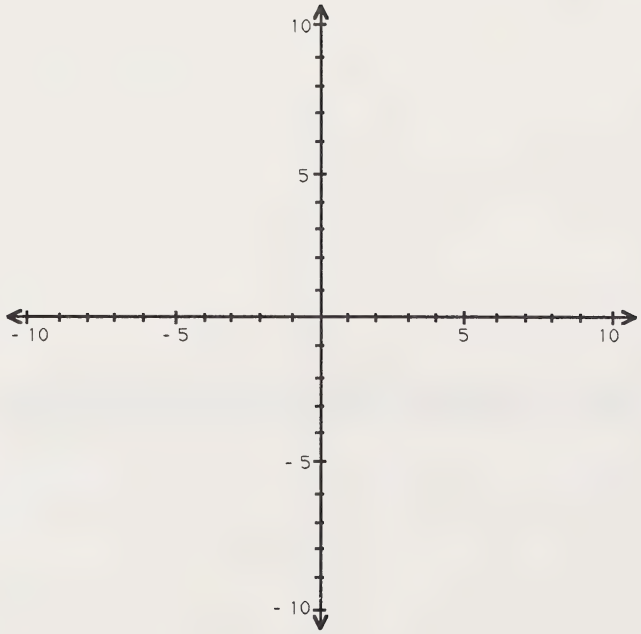




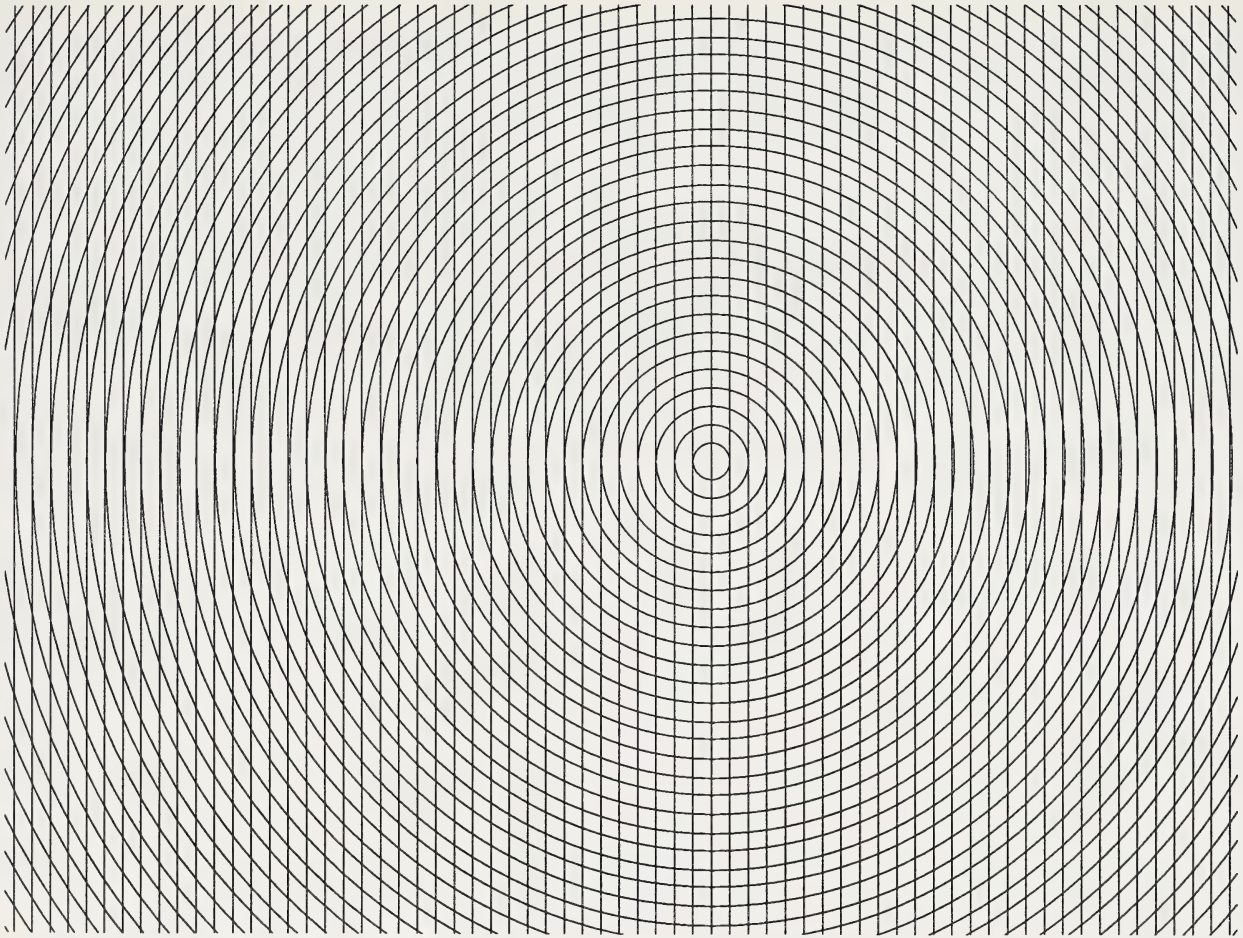

10
5
- 10
- 5
5
10
- 5
- 10

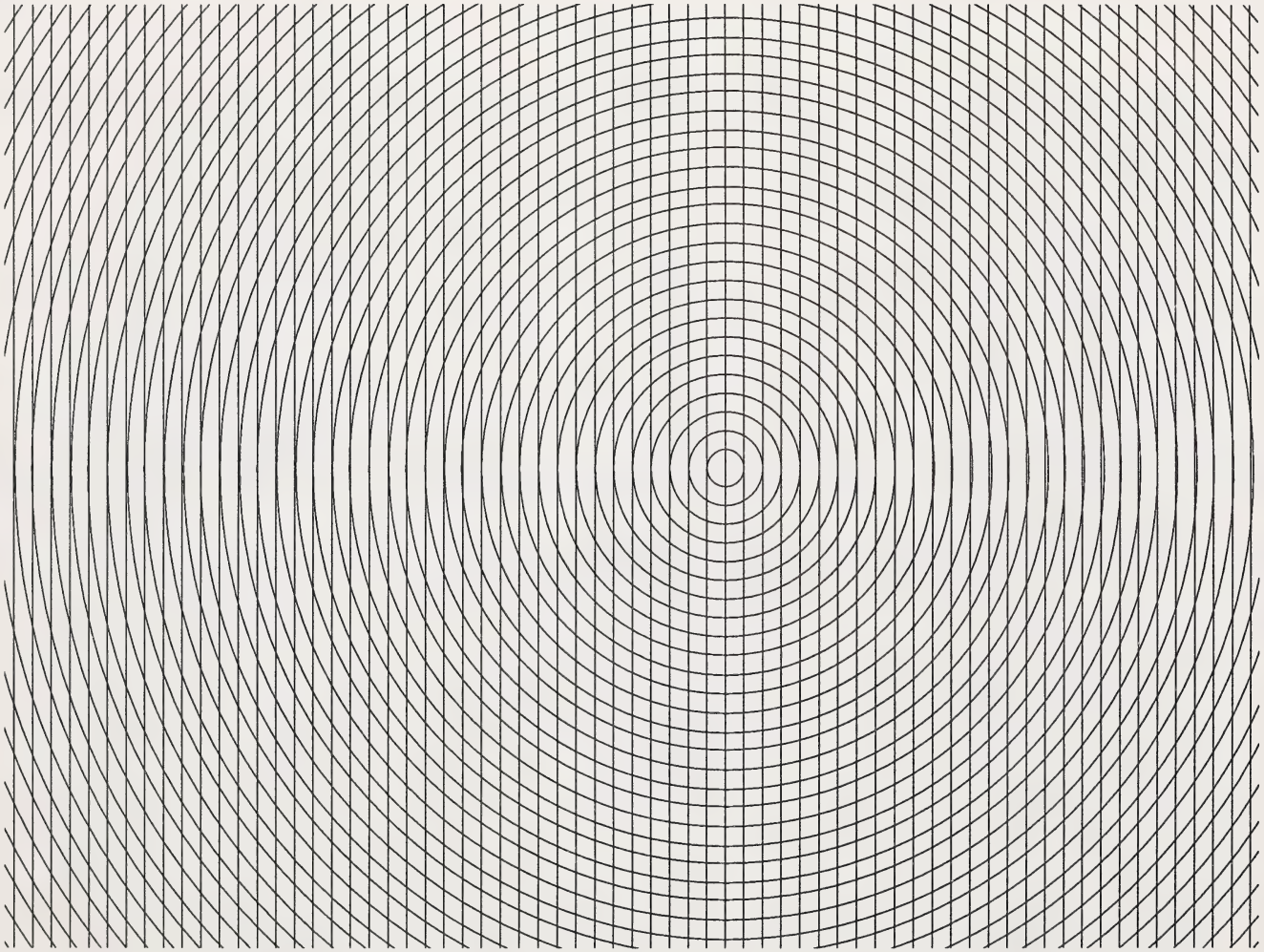

10
5
- 10
- 5
5
10
- 5
- 10

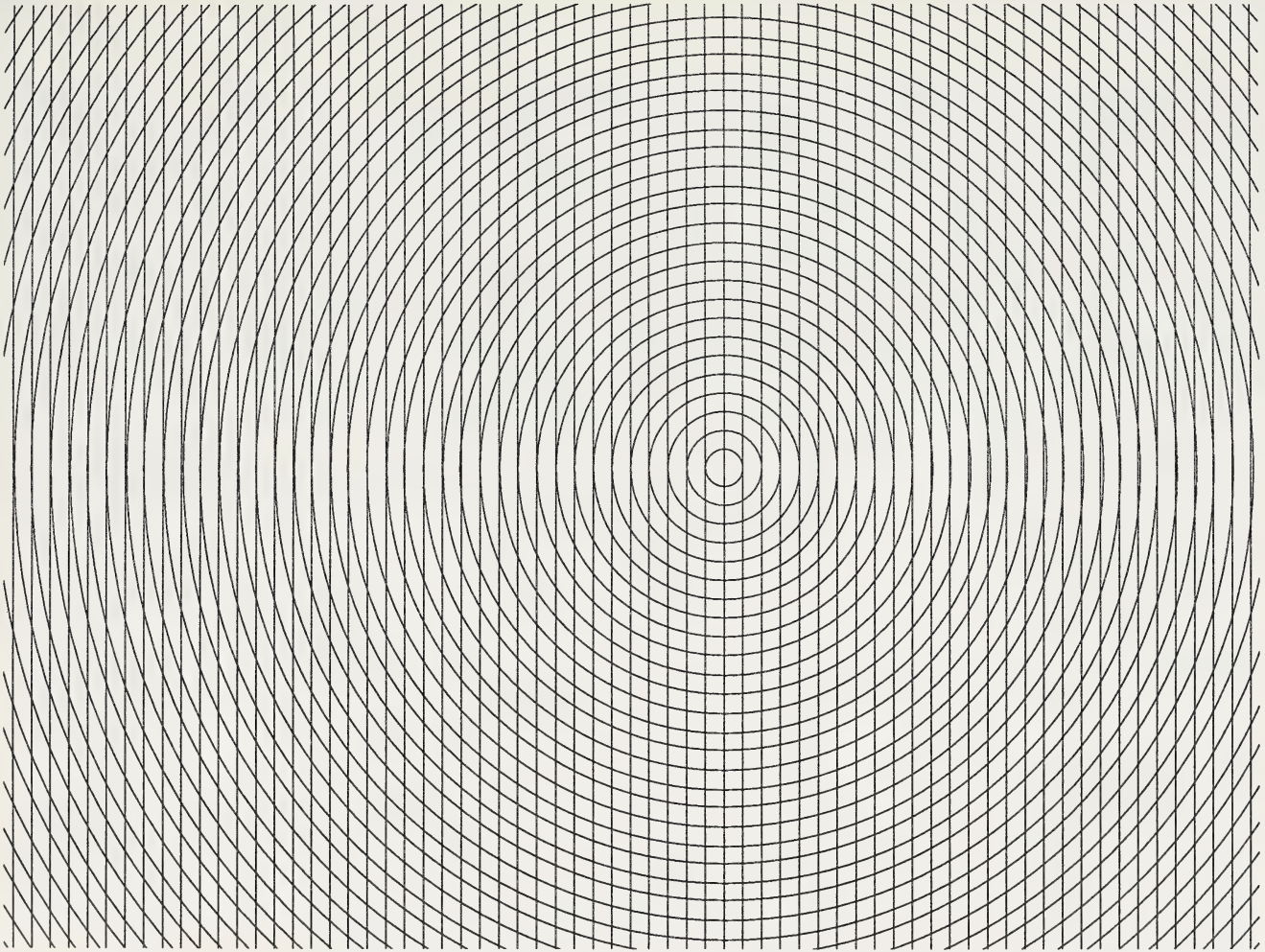

10
5
- 10
- 5
5
10
- 5
- 10

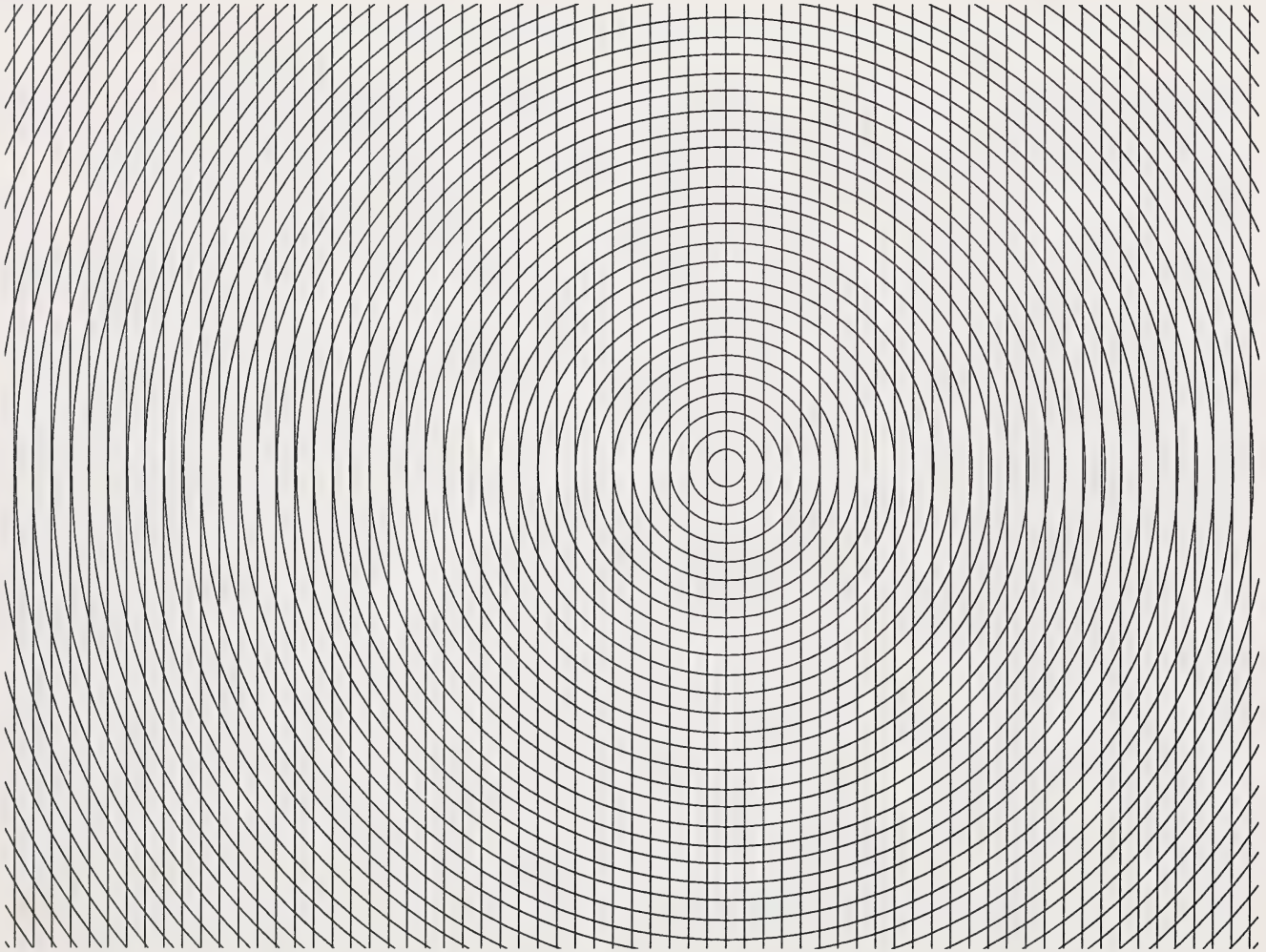

10
5
- 10
- 5
5
10
- 5
- 10

Circle-Line Graphs

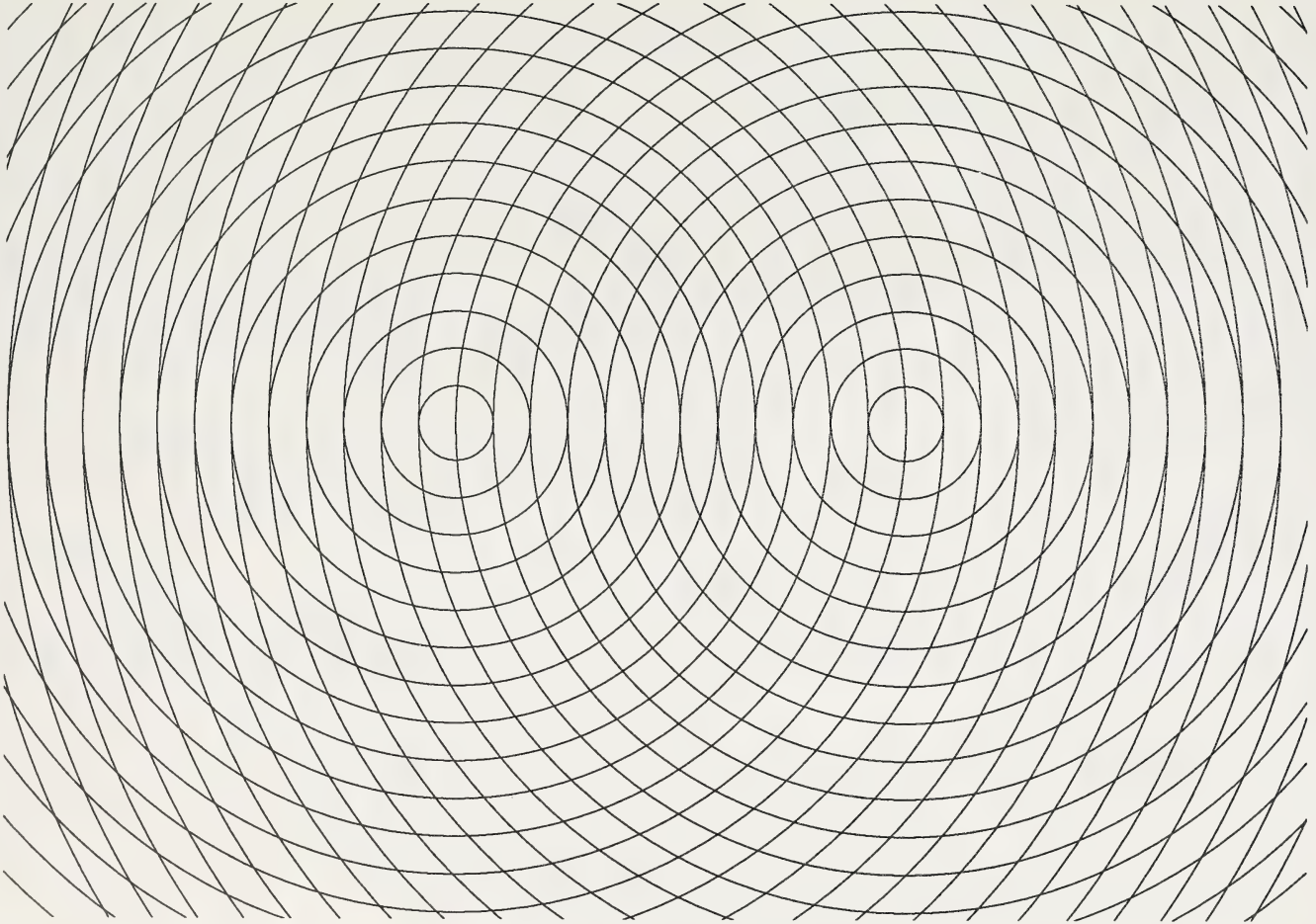


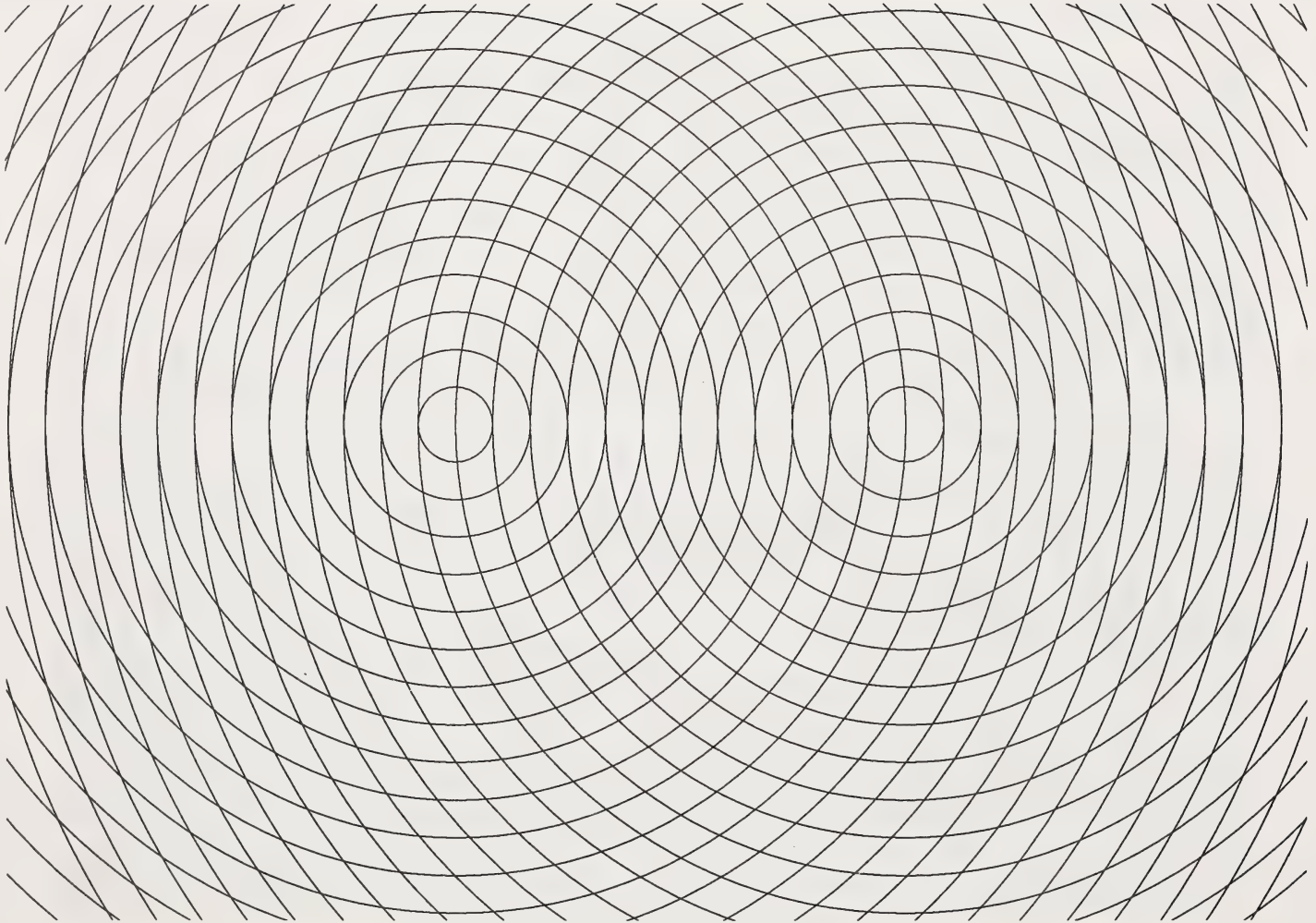


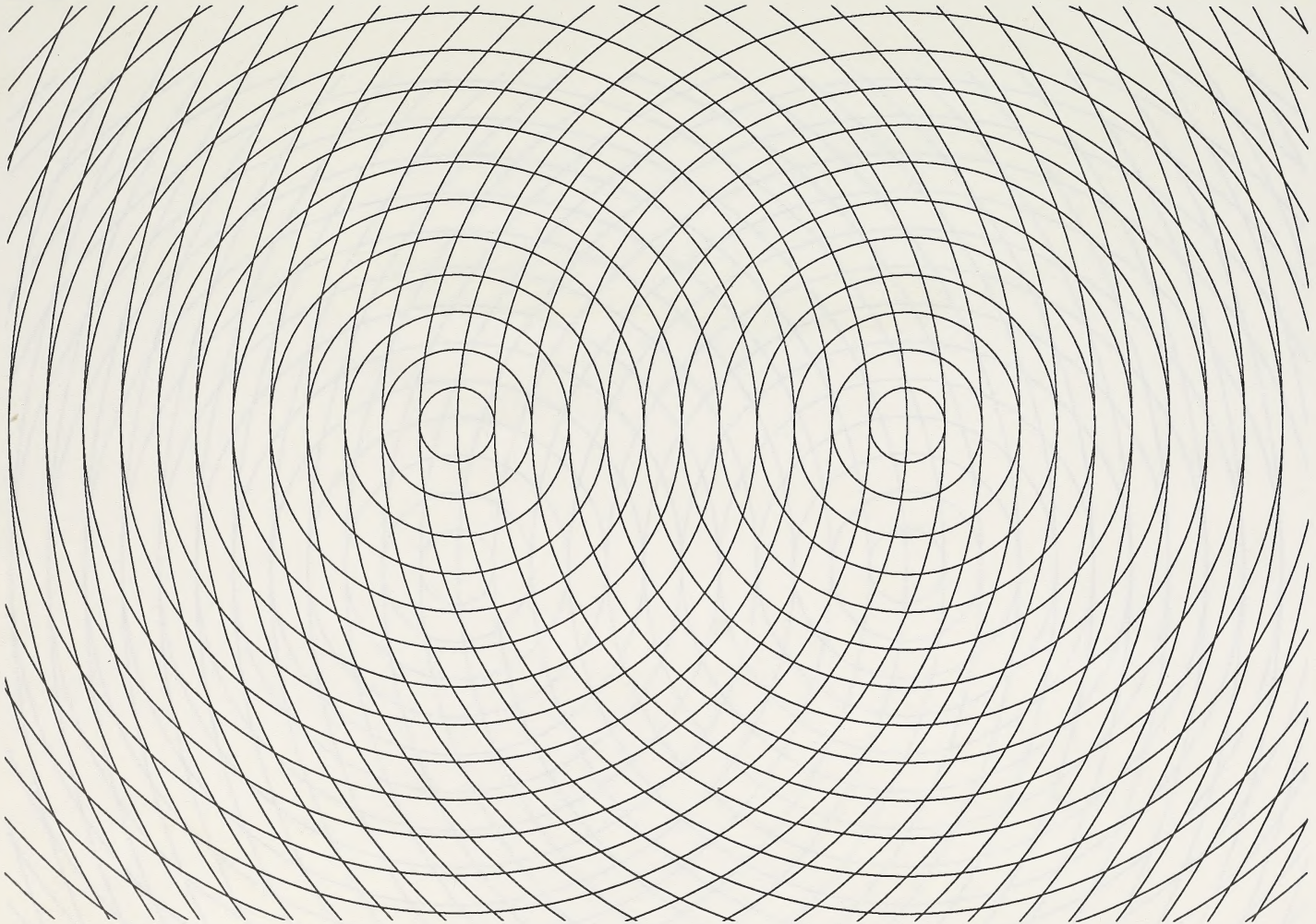


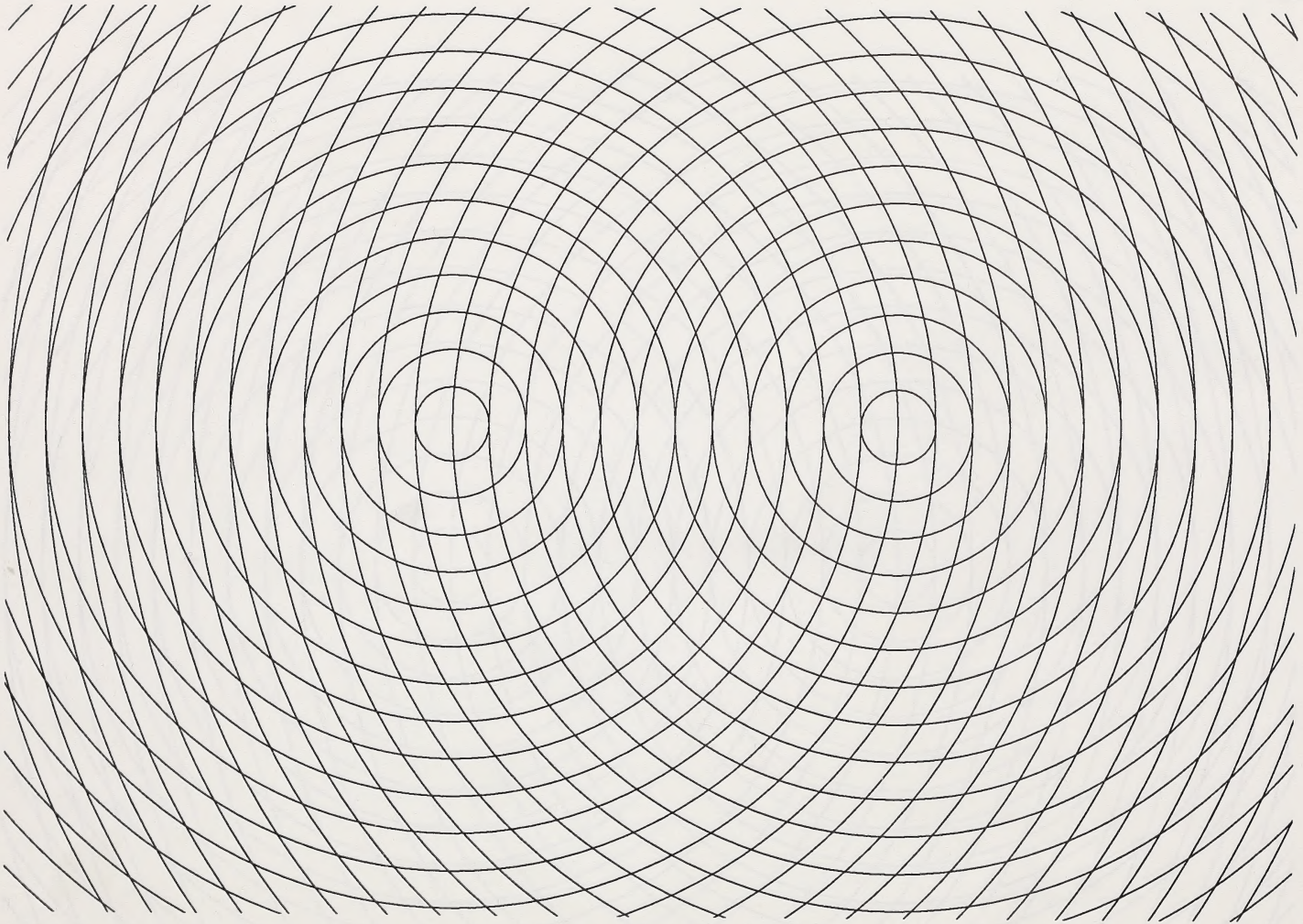


Double Concentric Circle Graphs









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